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INTERMEDIATE COURSE IN DIFFERENTIAL EQUATIONS

By

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PREFACE

Courses and books on differential equations are, for the most part, either elementary or advanced. The student who wishes to go beyond a first course is confronted with books which either are essentially the same as the one he has already studied or are far too advanced for his needs.

I have written this book with three main aims. The book is intended as a bridge for the gap between elementary courses and really advanced courses. It furnishes the student with further tools for constructing computable solutions for specific differential equations, in particular for linear equations in the neighborhood of regular singular points. Finally, it affords an introduction to several topics of importance in the classical theory.

It is my hope that this may serve as a useful textbook for a course following the usual one in elementary differential equations. It is advisable, mostly from the standpoint of mathematical maturity, that the student have had some advanced calculus. I have presented this material with equal success to students in pure mathematics and to students whose major interest was engineering, chemistry, physics, or statistics. It is conceivable that the book will prove adaptable for self-study for those of some maturity who find use for more than elementary differential equations in their own work, or who feel the desire to venture beyond their collegiate training in this field.

I wish to express my indebtedness to Professor Ruel V. Churchill of the University of Michigan, who read an early form of the manuscript, and to Professor Jack R. Britton of the University of Colorado, who read the manuscript and the proof sheets. Their comments and suggestions have helped me considerably. The aid and guidance which I received from the books of A. R. Forsyth, E. T. Whittaker, and G. N. Watson may well be imagined from the frequent references to them throughout.

This book is the outgrowth of my lecture notes, which were put out in dittoed form by the University of Michigan Secretarial Service in 1940 and in revised form in 1942.

EARL D. RAINVILLE

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CHAPTER I

LINEAR EQUATIONS OF THE SECOND ORDER

1. Introduction. This course is intended to do three things. It is to bring into play certain mathematical tools, naturally omitted in most elementary courses, which are useful in the solution of linear ordinary differential equations. It is also to furnish training which will aid a student going on to advanced work in the theory of differential equations to grasp more readily the aims, methods, and accomplishments of that theory. Finally, it is to present material much of which is of interest for its own sake.

We shall study many equations of importance in applied mathematics. On occasion we shall indicate properties of those equations which are not known (yet) to be of interest in applications. Although one of our aims is to develop methods and results which may be used in solving problems in engineering, physics, etc., we will not completely avoid interesting side lines, if they occur, merely because they bear at the moment the label "pure mathematics." Indeed, many of the equations which we shall meet are not only of interest in applications but are also fruitful, and therefore interesting, in pure mathematics. Anything we can learn about differential equations and their solutions may be of interest in applied mathematics, where we encounter differential equations so frequently. In pure mathematics the study of linear ordinary differential equations interests many students as a topic in itself and as a source of contacts with other branches of mathematics.

In the following pages we shall restrict ourselves to ordinary differential equations except when the contrary is specifically stipulated.

2. The General Linear Equation of the Second Order. The equation

$$(1) \qquad A_0(x)\frac{d^2w}{dx^2} + 2A_1(x)\frac{dw}{dx} + A_2(x)w = R_1(x)$$

and the equation

$$\frac{d^2w}{dx^2} + P(x)\frac{dw}{dx} + Q(x)w = R(x) \tag{2}$$

are equivalent, at least in regions where $A_0(x)$ is bounded from zero.* Sometimes we prefer to use the first of these equations, at other times the second. The labor introduced by the presence of an additional function in equation (1) is somewhat offset by the expected symmetry in form of the results obtained from that equation. In speaking of either of these equations, we may refer to it as the general linear equation of the second order.

We shall keep in mind that $A_0(x)$, $A_1(x)$, $P(x)$, etc., are functions of x, and we shall usually denote each by the appropriate single letter, A_0, A_1, A_2, R_1, P, Q, or R. Furthermore, we shall often use primes to denote differentiation where no confusion should arise. Thus: $y' = \dfrac{dy}{dx}$, $y'' = \dfrac{d^2y}{dx^2}$, etc.

Since we shall perform manipulations with the various functions involved, we need to make certain assumptions about those functions. We are not interested here in obtaining the weakest possible conditions which will insure the validity of our results. Complications can be avoided by assuming the continuity of each function and derivative which appears in our work. This we do. Beginning with Chapter IV, where we enter the less elementary phases of the subject, we shall be specific about the nature and behavior of the functions we use.

We shall show that any solution, other than $y \equiv 0$, of

$$y'' + Py' + Qy = 0 \tag{3}$$

leads at once to the general solution of equation (2).

Let y be a solution of equation (3), and put $w = yv$. Then

$$\begin{aligned} w' &= yv' + y'v \\ w'' &= yv'' + 2y'v' + y''v, \end{aligned}$$

and equation (2) becomes

$$yv'' + (2y' + Py)v' + (y'' + Py' + Qy)v = R,$$

---

* A function $f(x)$ is said to be bounded from zero in a region when there exists a $\delta > 0$ such that, for all x in the region, $|f(x)| > \delta$. It is vital that δ be independent of x.

in which y is a known function and v is to be determined. In view of equation (3), the above equation may be written

$$yv'' + (2y' + Py)v' = R,$$

a linear equation of the first order in v'. This is readily solved by using the integrating factor $ye^{\int P\,dx}$. We first multiply each member of our equation by this factor and thus obtain an exact equation, whose left member is the derivative of the product $y^2v'e^{\int P\,dx}$ and whose right member is a known function of x. Direct integration then yields an equation which we may solve for v' in terms of known functions and an arbitrary constant. Another integration gives us v itself in terms of known functions and two arbitrary constants. Finally, we obtain w from the relation $w = yv$. A similar process is illustrated in detail in the example worked out below.

We have thus reduced the problem of solving the general linear equation of the second order to the problem of finding a non-trivial ($y \not\equiv 0$) solution of the homogeneous equation

$$y'' + Py' + Qy = 0, \tag{3}$$

or its equivalent,

$$A_0y'' + 2A_1y' + A_2y = 0. \tag{4}$$

The student should note that the purpose of this section is the result stated in the preceding paragraph. The procedure outlined here is not necessarily the easiest in numerical problems. Several of the problems at the end of this section may be solved with less labor by using other methods learned in an elementary course, or by combining those methods with that of this section.

Example. Solve

$$(1 + x^2)y'' - 2xy' + 2y = x^3 + 3x,$$

using the fact that $y = x$ is a solution of the equation

$$(1 + x^2)y'' - 2xy' + 2y = 0.$$

Solution. We set $y = xv$, where v is now to be determined. At once $y' = xv' + v$ and $y'' = xv'' + 2v'$. If we substitute these expressions into our differential equation, we find that

$$(x^3 + x)v'' + 2v' = x^3 + 3x,$$

or

$$v'' + \frac{2}{x(x^2 + 1)}v' = \frac{x^2 + 3}{x^2 + 1}.$$

An integrating factor in this case is* $\exp\left[\int \frac{2\,dx}{x(x^2+1)}\right]$. Now, using partial fractions, we find that

$$\int \frac{2\,dx}{x(x^2+1)} = \int \frac{2\,dx}{x} - \int \frac{2x\,dx}{x^2+1} = 2\log x - \log(x^2+1),$$

and our integrating factor is seen to be

$$\exp\,[2\log x - \log(x^2+1)] = \exp\left(\log\frac{x^2}{x^2+1}\right) = \frac{x^2}{x^2+1}\cdot$$

The equation for v may now be written as

$$\frac{x^2}{x^2+1}v'' + \frac{2x}{(x^2+1)^2}v' = \frac{x^2(x^2+3)}{(x^2+1)^2}\cdot$$

But the left member of this equation is the derivative of $\frac{x^2v'}{x^2+1}\cdot$ Hence we have

$$\frac{x^2v'}{x^2+1} = \int \frac{x^2(x^2+3)}{(x^2+1)^2}\,dx = \frac{x^3}{x^2+1} + c_1,$$

where c_1 is an arbitrary constant. The integration is easily performed using integration by parts.

Next

$$v' = x + c_1\frac{x^2+1}{x^2} = x + c_1\left(1 + \frac{1}{x^2}\right).$$

Integrating once more, we find

$$v = \tfrac{1}{2}x^2 + c_1\left(x - \frac{1}{x}\right) + c_2,$$

which with $y = xv$ leads to the complete solution,

$$y = \tfrac{1}{2}x^3 + c_1(x^2 - 1) + c_2x.$$

The result may, of course, be checked by direct substitution into the original equation.

PROBLEMS

1. Solve

$$x(x-2)y'' - 2(x-1)y' + 2y = 2x^3 - 6x^2,$$

---

* We use the common notation $\exp\,(u) = e^u$ for convenience in printing. Note also that we need merely a single integrating factor, hence that the constant of integration inherent in the indefinite integral is of no use to us.

using the fact that $y = x^2$ is a solution of

$$x(x-2)y'' - 2(x-1)y' + 2y = 0.$$

Ans. $y = c_1x^2 + c_2(x-1) + x^3.$

2. Solve

$$(x-1)y'' - xy' + y = 1,$$

using the fact that $y = e^x$ satisfies the corresponding homogeneous equation.

Ans. $y = c_1x + c_2e^x + 1.$

3. Solve

$$y'' + 2y' + y = x^2 + 3x,$$

making use of the method of this section. Check the result directly.

Ans. $y = c_1e^{-x} + c_2xe^{-x} + x^2 - x.$

4. Solve

$$x^2y'' - x(x+2)y' + (x+2)y = -x^3 - x - 2,$$

making use of the substitution $y = xv$.

Ans. $y = c_1x + c_2xe^x + x^2 - 1.$

5. Solve the equation

$$x^2y'' + xy' - y = 0$$

by elementary methods. Then use your result to solve, by the method of this section, the equation

$$x^2y'' + xy' - y = -3x^2 - 1.$$

Ans. $y = c_1x + c_2x^{-1} + 1 - x^2.$

6. Use the particular solution $y = \sin x$ to solve completely the equation

$$\tan^2 x\, y'' - 2\tan x\, y' + (2 + \tan^2 x)y = 0.$$

Ans. $y = c_1 \sin x + c_2x \sin x.$

7. A particular solution of the equation

$$x^2y'' + (x^2 - 2x)y' - (2x^2 + x - 2)y = 0$$

is $y = xe^x$. Complete the solution of that equation.

Ans. $y = c_1xe^x + c_2xe^{-2x}.$

8. Show that you can always complete by elementary methods the solution of the equation

$$y'' - xQ(x)y' + Q(x)y = 0.$$

9. Solve the equation

$$x^3y'' - xy' + y = 0.$$

3. The Normal Form. In the equation

$$y'' + Py' + Qy = 0, \tag{3}$$

let us put

$$y = v \exp\left(-\tfrac{1}{2}\int P\,dx\right),$$

in order to remove the term involving the first derivative of the unknown function. The result is easily found to be

$$v'' + Iv = 0, \tag{5}$$

where

$$I = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2. \tag{6}$$

We say that we have transformed equation (3) into its *normal form,* equation (5). The function $I(x)$ in the normal form is often called an *invariant* of equation (3). For equation (4) the corresponding invariant is

$$I = A_0^{-2}\,(A_0A_2 - A_0A_1' + A_0'A_1 - A_1^2),$$

which may be derived in a manner similar to the above.

Let us now justify the name invariant for I. Suppose that the functions P, Q, P_1, and Q_1, are such that the equation

$$y'' + Py' + Qy = 0 \tag{3}$$

can be transformed into the equation

$$y_1'' + P_1y_1' + Q_1y_1 = 0 \tag{7}$$

by the substitution $y = fy_1$, where f is a function of x alone. We shall show that these equations have the same normal form.

If we actually perform the indicated transformation on (3), we find

$$fy_1'' + (2f' + Pf)y_1' + (f'' + Pf' + Qf)y_1 = 0,$$

from which

$$P_1 = 2\frac{f'}{f} + P,$$

$$Q_1 = \frac{f''}{f} + P\frac{f'}{f} + Q.$$

Let I_1 be the invariant in the normal form of equation (7), I that for equation (3). Then

$$\begin{aligned} I_1 &= Q_1 - \tfrac{1}{2}P_1' - \tfrac{1}{2}P_1^2 \\ &= \frac{f''}{f} + P\frac{f'}{f} + Q - \frac{ff'' - (f')^2}{f^2} - \tfrac{1}{2}P' - \frac{(f')^2}{f^2} - \frac{f'}{f}P - \tfrac{1}{4}P^2 \\ &= Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2 = I. \end{aligned}$$

Hence, the function I under consideration is unchanged or "invariant" under the transformation $y = fy_1$.

It is also easy to establish the fact that, if the two equations (3) and (7) have the same normal form, either can be transformed into the other by means of a transformation of the form $y = fy_1$. The transformation is

$$y = y_1 \exp [\tfrac{1}{2}\int (P_1 - P)\, dx].$$

We can, in general, actually perform the transformation of any linear differential equation of the second order into its normal form. Further, if we can obtain any non-trivial solution of the normal form of an equation, we can obtain the general solution of the original equation. Hence, two equations such as (3) and (7) with $I = I_1$ lead to precisely the same problem and will be called *equivalent*. Here the word equivalent is used in a technical sense.

EXAMPLE. Solve

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(a^2 + \frac{2}{x^2}\right)y = 0,$$

in which a is a constant.

Solution. We find that

$$I = a^2 + \frac{2}{x^2} - \frac{1}{2}\frac{2}{x^2} - \frac{1}{4}\frac{4}{x^2} = a^2.$$

Now, we already know the solution of

$$\frac{d^2v}{dx^2} + a^2v = 0$$

to be $v = c_1 \cos ax + c_2 \sin ax$. Hence we put

$$y = v \exp\left(\int \frac{dx}{x}\right) = xv,$$

and find the desired result to be

$$y = c_1x \cos ax + c_2x \sin ax,$$

where c_1 and c_2 are arbitrary constants.

PROBLEMS

1. Show that the equation

$$xy'' - 2(x - 1)y' + 2(x - 1)y = 0$$

is equivalent to the equation

$$v'' + v = 0,$$

and thus solve the former equation.

2. Obtain the normal form of the equation

$$x^2y'' + 4x^2y' + (4x^2 - 6)y = 0$$

and use it to solve the original equation.

3. Show that the equations

$$y'' - xy' + 2y = 0$$

and

$$w'' + xw' + 3w = 0$$

are equivalent. Find the relation between y and w.

4. Solve

$$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2(x^2 + 1)y = 0.$$

5. Find the value of $Q(x)$ which is such that the equation

$$\frac{d^2y}{dx^2} + a \frac{dy}{dx} + Qy = 0,$$

where a is a constant, may be transformed by a substitution $y = vf(x)$ into Bessel's equation,

$$\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} + \left(1 - \frac{n^2}{x^2}\right)v = 0,$$

and obtain the value of $f(x)$.

Ans. $Q(x) = \dfrac{a^2 + 4}{4} - \dfrac{4n^2 - 1}{4x^2}$.

$f(x) = c\sqrt{x} \exp\left(-\tfrac{1}{2}ax\right)$.

6. Obtain the normal form of the equation

$$x^2y'' + (x^2 - 2x)y' - (2x^2 + x - 2)y = 0$$

and thus solve Problem 7, page 5.

7. Apply the transformation $y = v \exp\left(-\frac{1}{2}\int P\,dx\right)$ to the equation

$$y'' + Py' + Qy = 0 \tag{3}$$

and thus derive the normal form,

$$v'' + Iv = 0, \tag{5}$$

in which $I = Q - \frac{1}{2}P' - \frac{1}{4}P^2$.

3. Obtain the invariant in the normal form of each of the following equations. These equations are among those to be studied later.

(*a*) $z(1-z)\dfrac{d^2w}{dz^2}+[\gamma-(\alpha+\beta+1)z]\dfrac{dw}{dz}-\alpha\beta w=0$; α, β, γ constants.

(*b*) $(1-x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+n(n+1)y=0$; n constant.

(*c*) $z\dfrac{d^2w}{dz^2}+(\gamma-z)\dfrac{dw}{dz}-\alpha w=0$; α, γ constants.

4. The Adjoint. In attempting to solve the equation

$$y''+Py'+Qy=0, \tag{3}$$

it is natural to search for an integrating factor $u(x)$. Assume, for the moment, the existence of such a factor. Then

$$uy''+uPy'+uQy=0$$

must be exact. Hence, it may be written

$$(uy')'+(hy)'=0,$$

where u and h are still to be determined. Performing the indicated differentiations, we have

$$uy''+(u'+h)y'+h'y=0.$$

We now compare coefficients in the two equations with identical first terms and arrive at the desired equations for u and h in terms of P and Q namely,

$$\begin{aligned} u'+h &= uP,\\ h' &= uQ. \end{aligned}$$

Eliminating h, we find an equation which must be satisfied by u, namely,

$$u''-Pu'+(Q-P')u=0. \tag{8}$$

This equation is called the *adjoint* of equation (3).

We can reverse the above steps and show that, if u is a solution of the adjoint of equation (3), then u is an integrating factor for equation (3). It follows that, if we can solve equation (8), we can reduce (3) to a linear differential equation of the first order. The reduced equation can then be solved at once by an elementary device.

We may push this procedure one more step and note that, if we can obtain a non-trivial solution of the adjoint of (8), then we can solve (8) and hence (3). Let us, then, find the adjoint of equation (8). It is to be found from (8) as that equation was found from (3). It is

$$w'' - (-P)w'' + [Q - P' - (-P)']w = 0,$$

or

$$w'' + Pw' + Qw = 0.$$

But this is equation (3) with merely a change of notation, so that no progress has been made. The failure to gain an advantage here has, however, brought to light the following interesting result:

The adjoint of the adjoint of equation (3) is equation (3).

When a process, such as the one immediately above, returns us to our starting point when performed twice, we call that process an *involution*. The operation of forming the adjoint of a linear differential equation of any order has been defined and studied elsewhere.* That operation is also an involution.

Next we prove a somewhat disheartening, but not at all astonishing, result. Let I be the invariant in the normal form of equation (3), and I_a the invariant in the normal form of its adjoint, (8). Then

$$I_a = Q - P' - \tfrac{1}{2}(-P)' - \tfrac{1}{4}(-P)^2 = I.$$

We have shown that equation (3) and its adjoint are equivalent.

It is worth noting that the adjoint of

$$(4) \qquad A_0y'' + 2A_1y' + A_2y = 0$$

is

$$A_0u'' + 2(A_0' - A_1)u' + (A_0'' - 2A_1' + A_2)u = 0,$$

as can be deduced by exactly the procedure followed above.

The adjoint of equation (4) can be put into a form much more easily remembered and more suggestive of generalization to equations of higher order. It is

$$(9) \qquad (A_0u)'' - (2A_1u)' + A_2u = 0.$$

Consider now the equation

$$(10) \qquad xy'' + y' + xy = 0,$$

---

* For example, A. R. Forsyth, *Theory of Differential Equations*, IV, 1902.

Bessel's equation of index zero, of which more will be said later. The adjoint of equation (10) is seen to have the same form as that equation. Such an equation is called self-adjoint. A slightly different concept, that of a self-adjoint boundary value problem in differential equations, has also been used considerably.

Next consider the equation

$$(11)\qquad y'' + \frac{1}{x}y' + y = 0,$$

obtained from equation (10) by dividing by x. The adjoint of equation (11) is

$$u'' - \frac{1}{x}u' + \left(1 + \frac{1}{x^2}\right)u = 0.$$

Hence equation (11) is not self-adjoint.

As a matter of fact, any linear equation of the second order can be put into self-adjoint form by merely introducing the proper factor. For the equation

$$(4)\qquad A_0y'' + 2A_1y' + A_2y = 0$$

the appropriate factor is

$$\frac{1}{A_0}\exp\left(2\int\frac{A_1}{A_0}dx\right).$$

PROBLEMS

1. Carry through the details in the derivation of the adjoint of the equation

$$A_0y'' + 2A_1y' + A_2y = 0.$$

2. Write down the adjoint of each of the following equations:

(*a*) $y'' + \frac{1}{x}y' - \left(1 + \frac{n^2}{x^2}\right)y = 0$; n constant.

(*b*) $x^2y'' - (x + 1)y' + 3y = 0$.

Ans. $x^2u'' + (5x + 1)u' + 6u = 0$.

(*c*) $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$, n constant.

(*d*) $y'' - \frac{2x}{1 - x^2}y' + \frac{n(n + 1)}{1 - x^2}y = 0$.

3. Follow the procedure of this section to define the adjoint of the general homogeneous linear differential equation of the third order. Show that the resultant operation is an involution.

4. Determine a change of dependent variable by which the equation

$$y'' + Py' + Qy = 0$$

may be transformed into its adjoint,

$$u'' - Pu' + (Q - P')u = 0.$$

5. Solve the equation

$$y'' - \left(2x + \frac{3}{x}\right)y' - 4y = 0$$

by obtaining by inspection a simple solution of its adjoint.

5. Change of the Independent Variable. In elementary courses there is usually some discussion of equations of the type

$$y'' + Py' + Qy = 0 \tag{3}$$

based upon a change of the independent variable from x to z. We may use the relations

$$y' = \frac{dz}{dx}\frac{dy}{dz},$$

$$y'' = \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} + \frac{d^2z}{dx^2}\frac{dy}{dz},$$

to obtain from equation (3) the alternative form:

$$\left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = 0. \tag{12}$$

The general idea behind this method of attack is that by some device we may be able to choose z so that equation (12) will reduce to a form we already know how to solve.

Let us determine conditions under which equation (12) becomes an equation with constant coefficients. It is certainly necessary that $\frac{1}{Q}\left(\frac{dz}{dx}\right)^2$ be constant. Thus we are led to choose the change of variable $z = c\int Q^{1/2}\,dx$, where c is a constant. With that choice in mind we see that (12) can become an equation with constant coefficients if, and only if, the ratio

$$\frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{Q}$$

is constant. From $z = c\int Q^{1/2}\,dx$, we find that

$$\frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{Q} = \frac{c}{2}\,\frac{Q' + 2PQ}{Q^{3/2}}.$$

We have proved the validity of the

TEST: *If in the equation*

$$y'' + Py' + Qy = 0 \tag{3}$$

the expression

$$\frac{Q' + 2PQ}{Q^{3/2}} \tag{13}$$

is found to be constant, then the change of independent variable $z = \int Q^{1/2}\,dx$ *will reduce equation (3) to an equation with constant coefficients. If the expression (13) is not constant, no change of independent variable alone will reduce equation (3) to an equation with constant coefficients.*

From the derivation of the test we see that the constant c is unessential.

We may, of course, find various tests similar to the one above by determining conditions under which it is possible to transform our original equation into a specific equation which we know how to solve. One example is furnished by Problem 7 below. The test resulting from that problem, however, is the same as the one we obtained above. Problem 9 introduces a different test. Others may be obtained by the student.

EXAMPLE. Solve:

$$y'' + \left(x - \frac{1}{x}\right)y' + x^2y = 0.$$

Solution. We first note that

$$\frac{Q' + 2PQ}{Q^{3/2}} = \frac{2x + 2x^2\left(x - \dfrac{1}{x}\right)}{x^3} = \frac{2x^3}{x^3} = 2.$$

Then we put $z = \int x\,dx = \frac{1}{2}x^2$. There is no advantage to be gained by introducing an added constant. Direct differentiation yields:

$$\frac{dz}{dx} = x,$$

$$\frac{dy}{dx} = x\frac{dy}{dz} = (2z)^{1/2}\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = x^2\frac{d^2y}{dz^2} + \frac{dy}{dz} = 2z\frac{d^2y}{dz^2} + \frac{dy}{dz}.$$

Hence the equation in y and z becomes

$$2z\frac{d^2y}{dz^2} + \frac{dy}{dz} + 2z\frac{dy}{dz} - \frac{dy}{dz} + 2zy = 0,$$

or

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} + y = 0.$$

From more elementary work we know that the general solution of this equation may be written with the aid of the roots $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ and $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ of the " auxiliary equation." That solution is

$$y = e^{-z/2}\left(c_1\cos\frac{\sqrt{3}}{2}z + c_2\sin\frac{\sqrt{3}}{2}z\right),$$

where c_1 and c_2 are arbitrary constants. Thus we arrive at the desired result,

$$y = e^{-x^2/4}\left(c_1\cos\frac{\sqrt{3}}{4}x^2 + c_2\sin\frac{\sqrt{3}}{4}x^2\right).$$

Finally we note that the method permits us to use instead of the above z any constant multiple of it. For instance, we could have used the change of independent variable $z = x^2$ rather than $z = \frac{1}{2}x^2$. The work would not be greatly changed.

PROBLEMS

1. Solve

$$y'' + \left(\frac{12x}{5} - \frac{1}{x}\right)y' + 4x^2y = 0.$$

Ans. $y = e^{-3x^2/5}\left(c_1\cos\frac{4x^2}{5} + c_2\sin\frac{4x^2}{5}\right).$

2. Solve

$$y'' + \tan x\, y' + \cos^2 x\, y = 0.$$

Ans. $y = c_1 \cos(\sin x) + c_2 \sin(\sin x).$

3. Solve

$$y'' + \left(12x^2 - \frac{2}{x}\right) y' + x^4 y = 0.$$

Ans. $y = c_1 \exp\left(\frac{-6 + \sqrt{35}}{3} x^3\right) + c_2 \exp\left(\frac{-6 - \sqrt{35}}{3} x^3\right).$

4. Solve

$$xy'' - 3y' + 16x^7 y = 0.$$

Ans. $y = c_1 \cos x^4 + c_2 \sin x^4.$

5. Solve

$$y'' + (2 \cos x + \tan x) y' + \cos^2 x\, y = 0.$$

Ans. $y = e^{-\sin x}(c_1 + c_2 \sin x).$

6. Introduce a symbol, say $4m$, for the expression (13). Let v be defined by the relation $v^2 = Q$. Then form and solve the corresponding auxiliary equation under the assumption that m is constant. Obtain what simplifications you can in the work of this section. Determine also, in terms of v and m, the general form of equations to which this method is applicable.

Ans. $y'' + \left(2mv - \frac{v'}{v}\right) y' + v^2 y = 0.$

7. Subject the equation

$$y'' + Py' + Qy = 0 \tag{3}$$

to the transformation $z = z(x)$ determined by setting

$$\frac{1}{Q}\left(\frac{dz}{dx}\right)^2 = z^2.$$

Then find the condition under which the resultant equation will become an equation of Euler type; that is,

$$\frac{1}{Q}\left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)$$

is to become a constant times z.

Ans. $\frac{Q' + 2PQ}{Q^{3/2}} = \text{a constant}.$

8. Prove that: A necessary and sufficient condition that the transformation $z = z(x)$ transform

$$y'' + Py' + Qy = 0 \tag{3}$$

into an exact equation is that $\frac{dz}{dx}$ be a solution of the adjoint,

$$u'' - Pu' + (Q - P')u = 0,$$

of equation (3).

9. Determine conditions under which a change of independent variable alone will transform equation (3) into Bessel's equation of index zero, which we shall solve later,

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + y = 0.$$

Ans. $\frac{Q' + 2PQ}{Q^{3/2}} \int Q^{1/2}\, dx = 2.$

10. Use the result in Problem 9 to show that the equation

$$y'' + \frac{1}{x}y' + 9x^4y = 0$$

can be transformed into Bessel's equation of index zero and perform the transformation directly.

6. The Schwarzian Derivative. We now consider the ratio of two essentially distinct (not proportional) non-trivial solutions of

$$v'' + Iv = 0.$$

Let the two solutions be v_1 and v_2, and put $v_1 = sv_2$. Then, since $v_1'' + Iv_1 = 0$ and $v_2'' + Iv_2 = 0$, we have, by substituting $v_1 = sv_2$ into the former equation, the result

$$v_2s'' + 2v_2's' = 0.$$

Differentiating this equation, we get

$$v_2s''' + 3v_2's'' + 2v_2''s' = 0,$$

or

$$s''' + 3\frac{v_2'}{v_2}s'' + 2\frac{v_2''}{v_2}s' = 0.$$

But $\frac{v_2'}{v_2} = -\frac{1}{2}\frac{s''}{s'}$, and $\frac{v_2''}{v_2} = -I$, so that we have

$$s''' - \frac{3}{2}\frac{(s'')^2}{s'} - 2Is' = 0,$$

or

$$\frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2 = 2I. \qquad (14)$$

The left member of equation (14) is called the *Schwarzian Derivative* of s with respect to x and is denoted by $\{s, x\}$.

If we can obtain a particular integral of equation (14), we can construct a non-trivial solution of $v'' + Iv = 0$ by means of the relation

$$\frac{v_2'}{v_2} = -\frac{1}{2}\frac{s''}{s'}.$$

From v_2 we can obtain, as indicated previously, the general solution of $v'' + Iv = 0$, or of any equation of which it is the normal form.

The Schwarzian Derivative may be considered independently of the way in which we have encountered it. We may, in other words, take

$$\{s, x\} = \frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2, \tag{15}$$

where primes denote differentiation with respect to x, as a definition of the symbol $\{s, x\}$. With this definition in mind we shall prove the following important theorem:

Theorem 1. If a, b, c, and d are constants and $ad - bc \neq 0$, then

$$\left\{\frac{as+b}{cs+d}, x\right\} = \{s, x\}.$$

The proof of Theorem 1 is somewhat simplified by using the result

$$\{s, x\} = \left(\frac{s''}{s'}\right)' - \frac{1}{2}\left(\frac{s''}{s'}\right)^2,$$

which is merely a simple rearrangement of (15). Now put

$$w = \frac{as+b}{cs+d}.$$

Then

$$w' = \frac{ad-bc}{(cs+d)^2}s',$$

and

$$w'' = \frac{ad-bc}{(cs+d)^2}s'' - \frac{2c(ad-bc)}{(cs+d)^3}(s')^2.$$

Since $ad - bc \neq 0$, we may divide by w' and find

$$\frac{w''}{w'} = \frac{s''}{s'} - \frac{2cs'}{cs+d}.$$

Hence,

$$\left(\frac{w''}{w'}\right)' = \left(\frac{s''}{s'}\right)' - \frac{2cs''}{cs+d} + \frac{2c^2(s')^2}{(cs+d)^2}.$$

Now

$$\frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{1}{2}\left(\frac{s''}{s'}\right)^2 - \frac{2cs''}{cs+d} + \frac{2c^2(s')^2}{(cs+d)^2},$$

so that

$$\{w, x\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \left(\frac{s''}{s'}\right)' - \frac{1}{2}\left(\frac{s''}{s'}\right)^2,$$

which is what we set out to prove.

In many branches of mathematics (theory of functions of a complex variable, geometry, etc.) the transformation

$$w = \frac{as+b}{cs+d}$$

with $ad - bc \neq 0$, called a non-singular *bilinear transformation* or a non-singular *linear fractional transformation*, is quite useful. We have shown that the Schwarzian Derivative is invariant under a non-singular bilinear transformation on the dependent variable.

Note that, if $ad - bc = 0$, then w is a constant and $\{w, x\}$ is undefined.

Theorem 1 leads at once to the conclusion that, if $ad - bc \neq 0$, then

$$\left\{\frac{ax+b}{cx+d}, x\right\} = \{x, x\} = 0.$$

We shall meet the Schwarzian Derivative often in the following pages.

PROBLEMS

1. Show that $\{x^m, x\} = -\frac{1}{2}(m^2 - 1)x^{-2}$.
2. Show that $\{\tan x, x\} = 2$.
3. Evaluate $\{\sin x, x\}$.

7. Equivalence under Change of Both Variables In later developments we shall find opportunity to use conditions under which

$$\frac{d^2y}{dx^2} + Iy = 0, \tag{16}$$

and

$$\frac{d^2v}{dz^2} + Jv = 0, \tag{17}$$

are equivalent in the sense that equation (16) may be transformed into equation (17) by changing both the dependent and independent variables. For the dependent variable we consider only transformations of the form $y = fv$, where f is a function of an independent variable and v is to be the new dependent variable.

If we first change the independent variable in (16) to z, we find that that equation becomes

$$\left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} + \frac{d^2z}{dx^2}\frac{dy}{dz} + Iy = 0. \tag{18}$$

We now desire conditions under which equations (17) and (18) are equivalent in the sense of Section 3. In that section we found those conditions to be: (a) the invariants in the normal forms of the two equations are equal, and (b) the dependent variables in the two equations are related by a simple equation given on page 7. Restated in our present notation, these conditions are

$$J = \left(\frac{dz}{dx}\right)^{-4}\left[\left(\frac{dz}{dx}\right)^2 I - \frac{1}{2}\left(\frac{dz}{dx}\right)^2 \frac{d^3z}{dx^3}\frac{dx}{dz} + \frac{dz}{dx}\frac{d^2z}{dx^2}\frac{dx}{dz}\frac{d^2z}{dx^2} - \frac{1}{4}\left(\frac{d^2z}{dx^2}\right)^2\right],$$

and

$$y = v \exp\left[-\frac{1}{2}\int \frac{d^2z}{dx^2}\left(\frac{dx}{dz}\right)^2 dz\right].$$

The first of the equations directly above may be written

$$\left(\frac{dz}{dx}\right)^2 J = I - \frac{1}{2}\frac{d^3z}{dx^3}\frac{dx}{dz} + \frac{3}{4}\left(\frac{d^2z}{dx^2}\right)^2\left(\frac{dx}{dz}\right)^2,$$

or

$$\left(\frac{dz}{dx}\right)^2 J = I - \tfrac{1}{2}\{z, x\}, \tag{19}$$

involving the Schwarzian Derivative of z with respect to x. Now

$$\frac{d}{dz}\left(\frac{dz}{dx}\right)^2 = 2\frac{dz}{dx}\frac{d^2z}{dx^2}\frac{dx}{dz} = 2\frac{d^2z}{dx^2},$$

so that

$$\frac{d^2z}{dx^2}\left(\frac{dx}{dz}\right)^2 = \frac{d^2z/dx^2}{(dz/dx)^2} = \frac{1}{2}\frac{d}{dz}\log\left(\frac{dz}{dx}\right)^2 = \frac{d}{dz}\log\frac{dz}{dx}.$$

Thus the second condition above, namely,

$$y = v \exp\left[-\frac{1}{2}\int\frac{d^2z}{dx^2}\left(\frac{dx}{dz}\right)^2 dz\right],$$

becomes

$$y = v \exp\left(-\frac{1}{2}\log\frac{dz}{dx}\right),$$

which may be written as follows:

$$(20) \qquad y = \left(\frac{dx}{dz}\right)^{1/2} v.$$

We have in equations (19) and (20) necessary and sufficient conditions for the type of equivalence we are now considering. Whenever we need to consider the equivalence under change of both variables of two equations not already in normal form, we can put them into normal form and then proceed as above.

There is another interesting way of looking at equation (19). Suppose that we are attempting to solve

$$\frac{d^2y}{dx^2} + Iy = 0.$$

We know $I(x)$. Let us choose $J(z)$ so that we know the solution of

$$\frac{d^2v}{dz^2} + Jv = 0.$$

Then, if we can find a particular solution of (19), considered as a differential equation determining z as a function of x, we can, by means of (20) and the known v, solve our original problem.

An obvious attempt in this direction is to choose $J \equiv 0$. Then we find that (19) reduces to $\{z, x\} = 2I$. Any solution of this equation leads at once to

$$y = \left(\frac{dx}{dz}\right)^{1/2} (c_1z + c_2),$$

the general solution of the original equation.

PROBLEMS

1. Prove that the equations

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - y = 0$$

and

$$4(1 - z)^2 \frac{d^2v}{dz^2} + 8z\frac{dv}{dz} + v = 0$$

are transformable into one another by means of a linear fractional transformation,

$$z = \frac{ax + b}{cx + d}.$$

Determine a, b, c, and d, and find the relation between y and v.

Ans. $z = \frac{x-1}{x}$. $y = x^{-1/2} e^x v$.

2. Show that the change of independent variable $z = \tan x$ permits the transformation of the equation

$$\frac{d^2y}{dx^2} + \tan^2 x\, y = 0$$

into the equation

$$(1 + z^2)^2 \frac{d^2v}{dz^2} + (z^2 - 1)v = 0,$$

and obtain the necessary relation between y and v.

Ans. $y = v \cos x$.

8. Resolution of Operators. There are times when the differential equation

$$A_0y'' + 2A_1y' + A_2y = 0$$

may be solved by factoring the operator $A_0\mathrm{D}^2 + 2A_1\mathrm{D} + A_2$, where $\mathrm{D} \equiv \frac{d}{dx}$, into the product $(p\mathrm{D} + q)(r\mathrm{D} + s)$. If the operator is thus factorable, we are led directly to the equation

$$(p\mathrm{D} + q)(r\mathrm{D} + s)y = 0,$$

which is readily solved by elementary devices. We can put $(rD + s)y = w$. Then $(pD + q)w = 0$. We solve a linear equation of the first order for w and then another of the same type for y, once w has been determined.

A student unfamiliar with the use of linear differential operators with variable coefficients should stop at this point and obtain some practice in handling the operators themselves. Note that, if A and B are linear differential operators, then except in special cases AB $\neq$ BA. For example, x^2D is defined as that operator which, acting upon a function y, yields $x^2 \dfrac{dy}{dx}\cdot$ The operator Dx^2, however, is defined as that operator which, acting upon y, yields the derivative of the product x^2y. That is,

$$(Dx^2)y = \frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + 2xy = (x^2D + 2x)y.$$

Hence $Dx^2 = x^2D + 2x \neq x^2D$. See also the first six problems at the end of this section.

Suppose that we have

$$(pD + q)(rD + s) = A_0D^2 + 2A_1D + A_2. \tag{21}$$

Now,

$$(pD + q)(rD + s) = prD^2 + (pr' + ps + qr)D + (ps' + qs),$$

so that our problem is reduced to determining p, q, r, and s from the equations

$$\begin{aligned} pr &= A_0, \\ pr' + ps + qr &= 2A_1, \\ ps' + qs &= A_2, \end{aligned}$$

where primes denote derivatives with respect to x.

In commenting upon the above equations, Forsyth has suggested* that we may take p and r to be known, subject to the restrictions $pr = A_0$. Then the other two equations serve to determine s and q. Let us eliminate q from the last two equations. We find

$$psr' + ps^2 - 2sA_1 - prs' + rA_2 = 0. \tag{22}$$

in which p, r, A_1, and A_2 are known functions and s is to be determined. Note, however, that the process we are now using is actually independent of whether p and r are considered to be known. This is a Riccati equation. For reasons which will become evident in Chapter II, we introduce a new dependent variable σ by means of the transformation

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* A. R. Forsyth, *A Treatise on Differential Equations*, 6th ed., 1933, 122.

$$s = -\frac{r\sigma'}{\sigma}.$$

Then

$$s' = -\frac{r'\sigma'}{\sigma} - \frac{r\sigma''}{\sigma} + \frac{r(\sigma')^2}{\sigma^2},$$

and equation (22) becomes

$$-\frac{prr'\sigma'}{\sigma} + \frac{pr^2(\sigma')^2}{\sigma^2} + \frac{2A_1r\sigma'}{\sigma} + \frac{prr'\sigma'}{\sigma} + \frac{pr^2\sigma''}{\sigma} - \frac{pr^2(\sigma')^2}{\sigma^2} + rA_2 = 0,$$

or

$$pr^2\sigma'' + 2A_1r\sigma' + rA_2\sigma = 0.$$

But $r \neq 0$ and $pr = A_0$, so that this equation is

$$A_0\sigma'' + 2A_1\sigma' + A_2\sigma = 0,$$

the equation we started out to solve. This is not startling. We have put the whole responsibility upon s. A knowledge of s will lead to the solution of our problem, and it is reasonable that the equation for the determination of s may be transformed into the original equation. More will be said about this in a discussion later in the chapter.

EXAMPLE. Solve

$$(x^2 - 3x + 2)\frac{d^2y}{dx^2} - (2x - 3)\frac{dy}{dx} + 2y = 0.$$

Solution. In an attempt to factor the operator involved, let us put

$$[(x - 1)D + f(x)][(x - 2)D + h(x)] = (x - 1)(x - 2)D^2 - (2x - 3)D + 2.$$

Since the coefficients of D^2 are equal, this leads to the identity

$$(x - 1)D + (x - 1)h(x)D + (x - 1)h'(x) + (x - 2)f(x)D + f(x)h(x) = -(2x - 3)D + 2.$$

Thus we have two equations to solve for $f(x)$ and $h(x)$, namely,

$$(x-1)+(x-1)h(x)+(x-2)f(x)=-2x+3$$

and

$$(x-1)h'(x)+f(x)h(x)=2.$$

We proceed naïvely with the assumption that we can determine $h(x)$ and $f(x)$ as linear functions of x. We have at present no justification for hoping that such a simple solution exists. The assumption we make does no harm, however, since, if we can determine such linear functions of x, their validity may be tested directly by substitution in the operational identity at the beginning of this solution.

Put $h(x)=ax+b$ and $f(x)=cx+d$. Then a, b, c, and d must satisfy the two identities

$$x-1+ax^2+bx-ax-b+cx^2+dx-2cx-2d\equiv -2x+3,$$
$$ax-a+acx^2+bcx+adx+bd\equiv 2,$$

or the six equations

$$\begin{aligned} a+c&=0,\\ 1+b-a+d-2c&=-2,\\ -1-b-2d&=3,\\ ac&=0,\\ a+bc+ad&=0,\\ -a+bd&=2. \end{aligned}$$

It is easy to show that this system of equations possesses the unique solution $a=0$, $b=-2$, $c=0$, $d=-1$.

Note, as a check, that

$$\begin{aligned}&[(x-1)\mathrm{D}-1][(x-2)\mathrm{D}-2]\\ &\quad=(x-1)(x-2)\mathrm{D}^2+(x-1)\mathrm{D}-2(x-1)\mathrm{D}-(x-2)\mathrm{D}+2\\ &\quad=(x^2-3x+2)\mathrm{D}^2-(2x-3)\mathrm{D}+2.\end{aligned}$$

Our original problem has been reduced to the problem of solving

$$[(x-1)\mathrm{D}-1][(x-2)\mathrm{D}-2]y=0.$$

Put $[(x-2)\mathrm{D}-2]y=w$. Then $[(x-1)\mathrm{D}-1]w=0$. This is

$$\frac{w'}{w}-\frac{1}{x-1}=0,$$

of which the solution is evidently $w = c_1(x - 1)$, where c_1 is an arbitrary constant. Next we have

$$(x - 2)y' - 2y = c_1(x - 1).$$

For this equation $(x - 2)^{-3}$ is an integrating factor. We find

$$(x - 2)^{-2}y' - 2(x - 2)^{-3}y = c_1(x - 1)(x - 2)^{-3},$$

from which

$$(x - 2)^{-2}y = c_1\int\frac{x - 1}{(x - 2)^3}\,dx = c_1\int\frac{1 + x - 2}{(x - 2)^3}\,dx,$$

or

$$(x - 2)^{-2}y = c_1[-\tfrac{1}{2}(x - 2)^{-2} - (x - 2)^{-1}] + c_2.$$

Finally

$$y = c_2(x - 2)^2 - c_1(x - 2) - \tfrac{1}{2}c_1,$$

the complete solution of the original equation. This result may be put in the form

$$y = c_2(x - 2)^2 + c_3(2x - 3).$$

Evidently a small exercise of the imagination originally would have induced us to try to find a quadratic solution of the problem. It is left as a problem for the student to show that the assumption $y = a_0x^2 + a_1x + a_2$ would have led to the above result. Of course, an attempt to solve the equation by the use of power series would also have led immediately to the same simple result. This, too, would furnish useful practice to the student familiar with series solutions.

PROBLEMS

1. Show that the operators $x^3\mathrm{D}$ and $\mathrm{D}x^3$ are not equivalent.

Ans. $\mathrm{D}x^3 = x^3\mathrm{D} + 3x^2 \neq x^3\mathrm{D}$.

2. If $\mathrm{A} = x\mathrm{D} + 1$ and $\mathrm{B} = \mathrm{D} - x$, form the two products of A and B.

Ans. $\mathrm{AB} = x\mathrm{D}^2 - (x^2 - 1)\mathrm{D} - 2x$.
$\mathrm{BA} = x\mathrm{D}^2 - (x^2 - 2)\mathrm{D} - x$.

3. Form the product $[(x - 2)\mathrm{D} - x][(x + 1)\mathrm{D} - 1]$.

Ans. $(x - 2)(x + 1)\mathrm{D}^2 - x(x + 1)\mathrm{D} + x$.

4. Form the product $[(x + 1)\mathrm{D} - 1][(x - 2)\mathrm{D} - x]$.

5. If $\mathrm{A} = x\mathrm{D}^2 - x\mathrm{D} + 2$ and $\mathrm{B} = \mathrm{D}^2 + x\mathrm{D} - \mathrm{D} + 1$, form the products AB and BA.

Ans. $\mathrm{AB} = x\mathrm{D}^4 + (x^2 - 2x)\mathrm{D}^3 - (x^2 - 4x - 2)\mathrm{D}^2 - 2\mathrm{D} + 2$.
$\mathrm{BA} = x\mathrm{D}^4 + (x^2 - 2x + 2)\mathrm{D}^3 - (x^2 - 3x + 1)\mathrm{D}^2 - \mathrm{D} + 2$.

6. If $A = (x - 1)D - 3$ and $B = (x - 1)D + 2$, form the products AB and BA.

Ans. $AB = BA = (x - 1)^2D^2 - 6.$

See also Problem 10 at the end of this chapter.

7. Solve the equation $[xD^2 + (2 - x)D - 1]y = 0.$

Ans. $xy = c_1e^x + c_2.$

8. Solve

$$(2x^2 + x - 1)\frac{d^2y}{dx^2} + 2(x - 2)\frac{dy}{dx} - 2y = 0.$$

Ans. $(x + 1)y = c_1(2x - 1)^2 + c_2$

9. Solve the equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

directly by elementary means, and then obtain a check by solving the equation by the method of this section.

10. Solve $[xD^2 - (x^2 - 2)D - x]y = 0.$

Ans. $xy = c_1\int e^{x^2/2}\,dx + c_2$, or

$$y = c_1x^{-1}\int_0^x e^{\beta^2/2}d\beta + c_2x^{-1}.$$

9. Solution in the Analytic Case. We mention briefly a general solution applying to, but not practical* for, most of the equations we shall meet in our present study.

If x is considered as a complex variable, then a function $f(x)$ is called analytic at the point $x = x_0$ if, and only if,

$$\lim_{h\to 0}\frac{f(x + h) - f(x)}{h}$$

exists for each x in some region including x_0. Existence here implies independence of the manner in which the complex variable h approaches zero. One of the many equivalent definitions is: $f(x)$ is analytic at $x = x_0$ if, and only if, it has a convergent power series representation in some neighborhood of that point. These matters may be taken as the starting points of one of the major fields of mathematics, and we shall confine ourselves to the brief descriptive account in Chapter III.

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* See the last paragraph of this section.

Suppose that $I(x)$ is analytic at $x = x_0$. We shall present a solution of the equation

$$\frac{d^2y}{dx^2} + I(x)y = 0$$

and shall attempt to make the result plausible, which is all we can hope to do without some use of the theory of functions of a complex variable.

Let a_0 and a_1 be arbitrary constants, and define $V_n(x)$; $n = 0, 1, 2, \cdots$, by

$$V_0(x) = a_0 + a_1(x - x_0)$$

and, for $n \geqq 1$,

$$V_n(x) = \int_{x_0}^{x} (\sigma - x)I(\sigma)V_{n-1}(\sigma)d\sigma.$$

Now $V_0'(x) = a_1$, $V_0''(x) = 0$, and, for $n \geqq 1$,

$$V_n'(x) = -\int_{x_0}^{x} I(\sigma)V_{n-1}(\sigma)d\sigma,$$

$$V_n''(x) = -I(x)V_{n-1}(x).$$

Consider the function

$$y(x) = \sum_{n=0}^{\infty} V_n(x).$$

Formally we have

$$\begin{aligned}\frac{d^2y}{dx^2} + Iy &= \sum_{n=0}^{\infty} V_n''(x) + \sum_{n=0}^{\infty} I(x)V_n(x)\\ &= -\sum_{n=1}^{\infty} I(x)V_{n-1}(x) + \sum_{n=0}^{\infty} I(x)V_n(x)\\ &= 0.\end{aligned}$$

That $y(x)$ so defined is actually a solution is proved in Whittaker and Watson, *Modern Analysis*, 4th ed., 1940, 195. There the uniform convergence of the series for $y(x)$ is demonstrated and the various steps in the proof are justified. Indeed, subject to the requirement of analyticity of the solution and satisfaction of the boundary conditions $y(x_0) = a_0$, $y'(x_0) = a_1$, the above solution is proved to be unique.

When considering using the above solution it is well to keep in mind that $V_n(x)$, the general term of the series, is given in the form of an n-fold integral.

PROBLEM

1. Using $a_0 = 0$, $a_1 = 1$, $x_0 = 0$, find a solution of

$$\frac{d^2y}{dx^2} + xy = 0.$$

10. Summary and Discussion of Results. In the preceding sections we have considered a series of elementary attacks on the problem of solving linear differential equations of the second order. Essentially our procedure has been to obtain various equations equivalent to the original problem. It may well be that, although no real simplification of the problem has been accomplished, our own mental behavior is such that it is easier for us to arrive at a solution when a problem is presented in one form rather than in another.

In Chapter II we shall be concerned with Riccati equations. Any linear differential equation of the second order may be transformed into a Riccati equation, so that there we shall meet still another form equivalent to our original problem.

We have also considered, particularly in Section 5, criteria which determine for us immediately whether certain devices for solving an equation will work for the particular equation. In more elementary courses this type of attack has already been introduced, notably for exact differential equations of the first order and the first degree and for equations reducible to those by means of an integrating factor. A good summary of integrating factors for certain types of equations is included in the *Smithsonian Mathematical Formulae and Tables of Elliptic Functions,* 1922, 163–164.

In Section 9 we glanced at a solution applicable to a fairly general class of second-order linear equations. Unfortunately, the form of that solution is such as to make it definitely unpopular in many specific applications.

A particularly powerful method of attack has not been treated in this chapter but will receive a good deal of attention in later chapters. This is the use of power series. We shall, indeed, find that very many of the non-elementary functions useful in applications arise as solutions of differential equations which may be solved by means of power series in the independent variable. It appears also that many of these functions are special cases or "limiting cases" of a solution in power series of the famous hypergeometric equation of Gauss. To this important equation and its solution by series we shall devote much of our time.

MISCELLANEOUS PROBLEMS

1. Solve by two distinct methods:

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = 0.$$

2. Solve

$$(x^2 - a^2)\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + 12y = 0.$$

3. Solve, using a simple trigonometric particular solution,

$$\sin^2 x\frac{d^2y}{dx^2} = 2y.$$

4. The equations

$$y'' + P_1y' + Q_1y = 0,$$
$$y'' + P_2y' + Q_2y = 0,$$

are assumed to have a solution in common. What relation does this impose upon the functions P_1, P_2, Q_1, and Q_2? Is this necessary relation also sufficient to imply that the two equations do have a common solution? Complete the solution of each equation under the original assumption.

5. Derive the expression for the invariant in the normal form of the equation

$$A_0y'' + 2A_1y' + A_2y = 0.$$

Do this directly and also by using the known result for equation (3),

$$y'' + Py' + Qy = 0.$$

6. Discuss a possible use of the expression

$$I = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2$$

if the coefficients in the original equation are constant. *Hint:* Consider the discriminant of the auxiliary quadratic equation.

7. Prove that: In the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \tag{3}$$

the expression

$$\frac{(dQ/dx) + 2PQ}{Q^{3/2}} \tag{13}$$

is invariant under a transformation $z = z(x)$ on the independent variable alone. That is, change the independent variable in equation (3) to z, and let the transformed equation (12), page 12, be written as

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = 0.$$

Then show that

$$\frac{(dQ_1/dz) + 2P_1Q_1}{Q_1^{3/2}} = \frac{(dQ/dx) + 2PQ}{Q^{3/2}}.$$

8. With the terminology of Problem 7, show that the expression

$$\int Q^{1/2}\,dx$$

is also invariant under a transformation of the independent variable. Note that the invariance of the expressions involved in this and the preceding problem is suggested by the results of Problems 7 and 9 of Section 5.

9. If $\mathrm{A} = 2x\mathrm{D} + 1$ and $\mathrm{B} = x\mathrm{D} + 3$, form the products AB and BA.

Ans. $\mathrm{AB} = \mathrm{BA} = 2x^2\mathrm{D}^2 + 9x\mathrm{D} + 3$.

10. Prove: If $\mathrm{A} = (a_1x + b_1)\mathrm{D} + c_1x + d_1$ and $\mathrm{B} = (a_2x + b_2)\mathrm{D} + c_2x + d_2$, where a_1, a_2, b_1, b_2, c_1, c_2, d_1, and d_2 are constants, then a necessary and sufficient condition that A and B be commutative (that is, that $\mathrm{AB} = \mathrm{BA}$) is that

$$\mathrm{A} = k_1\mathrm{B} + k_2,$$

where k_1 and k_2 are constants.

CHAPTER II

RICCATI EQUATIONS

11. Introduction. What is usually known as the generalized Riccati* equation is

$$\frac{dy}{dx} = A_0(x) + A_1(x)y + A_2(x)y^2. \tag{23}$$

We shall call (23) a *Riccati equation,* reserving modifying terms for the special cases.

There are at least two approaches which lead naturally to a study of equation (23). On the one hand every second-order linear differential equation can be transformed into a Riccati equation. The other approach is through the study of the general equation of the first order and the first degree,

$$\frac{dy}{dx} = f(x, y). \tag{24}$$

If $f(x, y)$ is linear in y, equation (24) reduces to a form we already know how to solve. In the ordinary course of events the next case† to consider would be equation (23).

The special Riccati problem is the solution of the equation $y' + by^2 = cx^m$, where b, c, and m are constants. This equation is discussed in Forsyth's treatise,‡ where particular attention is paid to its integrability in " finite terms " and its relation to Bessel's equation.

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[*] Jacopo Francesco, Count Riccati (1676–1754), published his study of what is now known as the special Riccati problem in the *Acta eruditorum,* 1724.

[†] There is, however, a stronger reason for treating this particular case of equation (24). In the function-theoretic treatment of equation (24) it is found that the generalized Riccati equation is the most general form of (24) compatible with the conditions that $f(x, y)$ be rational in y and that the general solution of the equation have no movable singularities other than poles. In the theory of differential equations this matter is gone into in detail. Here we must omit the topic.

[‡] A. R. Forsyth, *A Treatise on Differential Equations,* 6th ed., 1933, 188–197.

With regard to the relations between Riccati equations and linear equations of the second order, consider the effect of the change of dependent variable

$$y = \exp\left(-\int u\,dx\right)$$

upon the equation $y'' + Py' + Qy = 0$. Since

$$y' = -u \exp\left(-\int u\,dx\right)$$

and

$$y'' = (u^2 - u')\exp\left(-\int u\,dx\right),$$

we see that the transformed equation is $u' = Q - Pu + u^2$, a Riccati equation.

If, on the other hand, we apply the transformation

$$y = -\frac{w'}{wA_2}$$

to the equation $y' = A_0 + A_1y + A_2y^2$, we find that

$$A_2w'' - (A_2' + A_1A_2)w' + A_2^2A_0w = 0.$$

We have thus shown that the generalized Riccati equation leads to the same problem as the linear equation of the second order.

PROBLEMS

1. Transform each of the following equations into a Riccati equation:

(*a*) $xy'' + y' + xy = 0$.
(*b*) $(1 - x^2)y'' - 2xy' + 6y = 0$.
(*c*) $v'' + Iv = 0$.

2. Transform each of the following equations into a linear equation of the second order:

(*a*) $y' = 1 - x^2 + y^2$.
(*b*) $y' = 1 + x^2y - xy^2$.
(*c*) $xy' = 1 + y + xy^2$.

12. General Solution When a Particular Solution is Known. Suppose that we know y_1, a particular solution of

$$y' = A_0 + A_1y + A_2y^2. \tag{23}$$

Because of the relation of this equation to the linear equation of the second order, we expect to be able to find the general solution of (23) from y_1. It is logical to go back to the equivalent linear equation to determine how to use the known solution.

The transformation $y = -\dfrac{w'}{wA_2}$ will carry (23) over into a linear equation. Corresponding to the solution y_1 of equation (23) there will be a solution w_1 of the linear equation. We define w_1 by the relation $y_1 = -\dfrac{w_1'}{w_1 A_2}$. Previous experience dictates that we put $w = w_1\sigma$, where σ is to be a new dependent variable in the linear equation. This leads at once to $y = y_1 - \dfrac{\sigma'}{\sigma A_2}$, which suggests that in the Riccati equation we introduce a new dependent variable t by the substitution $y = y_1 + t$. Finally, the equation in t is found to be reducible to a linear equation of the first order in $v = \dfrac{1}{t}$. The details are left to the student. We proceed to use the resultant method.

Put $y = y_1 + \dfrac{1}{v}$. Then

$$y_1' - \frac{v'}{v^2} = A_0 + A_1 y_1 + \frac{A_1}{v} + A_2 y_1^2 + \frac{2A_2 y_1}{v} + \frac{A_2}{v^2}.$$

But $y_1' = A_0 + A_1 y_1 + A_2 y_1^2$, since y_1 satisfies (23). Hence

$$v' + (A_1 + 2A_2 y_1)v = -A_2.$$

Using the integrating factor $T(x) = \exp\left[\int (A_1 + 2A_2 y_1)\, dx\right]$ we are led at once to the result

$$v = \frac{b - \int A_2 T(x)\, dx}{T(x)},$$

where b is an arbitrary constant. Returning to the transformation, we find that

$$y = \frac{by_1 + T(x) - y_1 \int A_2 T(x)\, dx}{b - \int A_2 T(x)\, dx},$$

which is the general solution of (23) in terms of a particular solution y_1. In actual practice it may seem unwise to use such an unwieldy formula as that directly above. Instead, we often carry out the process itself.

From the above discussion it is seen that the general solution of (23) may always be expressed bilinearly in terms of an arbitrary constant; that is

$$y = \frac{bf_1(x) + f_2(x)}{bf_3(x) + f_4(x)}. \tag{25}$$

This is true because, if the general solution is known in any form whatever, then a specialization of it yields a particular solution from which (25) follows at once.

We shall have more to say, later in this chapter, about the results derivable from the knowledge of particular solutions. There, also, an interesting theorem relating any four solutions of a Riccati equation will be discussed. At the moment we digress.

Example. Solve the equation

$$y' = x - 2 + xy + y^2,$$

being given that $y = 1 - x$ is a particular solution.

Solution. First we put $y = 1 - x + \frac{1}{v}$ and find that our equation reduces to

$$v' - (x - 2)v = -1.$$

From the elementary theory we determine the integrating factor $\exp[-\frac{1}{2}(x-2)^2]$. Then we have

$$\exp[-\tfrac{1}{2}(x-2)^2]v' - (x-2)\exp[-\tfrac{1}{2}(x-2)^2]v = -\exp[-\tfrac{1}{2}(x-2)^2],$$

from which

$$\exp[-\tfrac{1}{2}(x-2)^2]v = -\int \exp[-\tfrac{1}{2}(x-2)^2]\,dx + c,$$

where c is an arbitrary constant. We could now write down y in terms of x, but the form might be confusing because of the use of x both as a variable of integration and as the independent variable in our problem. This situation arises, of course, because $\int \exp[-\frac{1}{2}(x-2)^2]\,dx$ is not expressible by means of "elementary functions" in a finite number of steps.

In order to avoid this possible confusion it seems desirable to introduce a definite integral. We may choose for the lower limit whatever constant we wish. Let us write

$$\exp\left[-\tfrac{1}{2}(x-2)^2\right]v = -\int_2^x \exp\left[-\tfrac{1}{2}(\beta-2)^2\right]d\beta + c.$$

Now we are tempted to introduce a new variable of integration by making the substitution $\beta - 2 = \sqrt{2}\alpha$. We do so. Then we have

$$\exp\left[-\tfrac{1}{2}(x-2)^2\right]v = -\sqrt{2}\int_0^{(x-2)/\sqrt{2}} e^{-\alpha^2}\,d\alpha + c.$$

Here we have a definite integral which occurs frequently in the subjects of statistics, probability, heat conduction, and elsewhere. Indeed, the function which we shall define by

$$P(w) = \frac{2}{\sqrt{\pi}}\int_0^w e^{-\alpha^2}\,d\alpha$$

has been tabulated* under the name "the probability integral."

With the notation above we may write

$$\exp\left[-\tfrac{1}{2}(x-2)^2\right]v = -\sqrt{\frac{\pi}{2}}\,P\left(\frac{x-2}{\sqrt{2}}\right) + c.$$

Hence the solution to our problem is

$$y = 1 - x + \frac{\exp\left[-\tfrac{1}{2}(x-2)^2\right]}{c - \sqrt{\dfrac{\pi}{2}}\,P\left(\dfrac{x-2}{\sqrt{2}}\right)}.$$

PROBLEMS

1. Solve the equation

$$y' = x^2 - 2 + 2xy + y^2,$$

using the particular solution $y = 1 - x$.

Ans. $y = 1 - x + \dfrac{2e^{2x}}{c - e^{2x}}$.

2. Solve the equation

$$y' = x^5 + x^2 + 2x - y - xy^2.$$

Ans. $y = x^2 + \dfrac{\exp\left(-\tfrac{1}{2}x^4 - x\right)}{c + \displaystyle\int_0^x \beta\exp\left(-\tfrac{1}{2}\beta^4 - \beta\right)d\beta}$.

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* See B. O. Pierce, *A Short Table of Integrals*, 1929. The function $P(w)$ is sometimes called the error function and is then written *erf* (w).

3. Use the particular solution $y = 1$ to solve the equation

$$\frac{dy}{dx} = x - 1 + y - xy^2.$$

4. Solve the equation

$$(1 - x^2)\frac{dy}{dx} = 1 - y^2$$

(*a*) By elementary means.
(*b*) Using the particular solution $y = 1$.
(*c*) Using the particular solution $y = -1$.
(*d*) Using the particular solution $y = x$.

5. Show from those solutions that the four forms you obtain in Problem 4 are equivalent.

6. Use the particular solution $y = \sin x$ in completing the solution of the equation

$$\frac{dy}{dx} = \cos x - \sin x\, y + y^2.$$

Ans. $y = \sin x + \dfrac{e^{-\cos x}}{c - \int_0^x e^{-\cos \beta}\, d\beta}$.

13. Polynomial Solutions. The preceding section makes evident the value of a particular solution. At times we obtain what are often called "inspection" solutions. For example, it is easily seen that the equation

$$y' = 1 - x^2 + y^2$$

has the particular solution $y = x$. From this the general solution would follow at once.

If the equation has only polynomials in x for coefficients, we are tempted to search for a particular solution in the form of a polynomial in x. As usual in mathematics we do not like to face the prospect of a blind hit-or-miss process. Hence, we look for a method for restricting the polynomials we need test as possible solutions of our equation.*

First, consider the transformation $y = \dfrac{u}{A_2}$. This changes the equation

$$y' = A_0 + A_1y + A_2y^2 \tag{23}$$

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* The material in Sections 13, 14, and 16 appeared in two papers by the author in the *Amer. Math. Monthly*, 43, 1936, 473–476, and 48, 1941, 519–521. Those papers contain no additional results.

into

$$\frac{u'}{A_2} - \frac{uA_2'}{A_2^2} = A_0 + \frac{A_1u}{A_2} + \frac{u^2}{A_2}$$

or

$$u' = B_0 + B_1u + u^2, \tag{26}$$

where $B_0 = A_0A_2$ and $B_1 = A_1 + \frac{A_2'}{A_2}$. We shall restrict ourselves in this section to those Riccati equations of the form of (26) with B_0 and B_1 polynomials* in x. We shall show that in general only one or two perfectly well-defined polynomials need be tested as possible solutions. No polynomials other than these can be solutions.

Definition 1. By the symbol $[\sqrt{P(x)}]$, where $P(x)$ is a polynomial of even degree, we shall mean the polynomial part of the expansion of $\sqrt{P(x)}$ in a series of descending integral powers of x. In order to insure uniqueness of the result, we choose from the two possible coefficients (permitted to be complex) of the highest power of x in $\sqrt{P(x)}$ that coefficient with the smaller positive amplitude.

It seems easiest to obtain $[\sqrt{P(x)}]$ by an old square-root-extraction process of elementary arithmetic. This is done in the numerical example at the end of this section. More advanced methods for the expansion of a function into descending powers of a variable are, of course, available.

With the above notation we may more easily state

Theorem 2. If in the equation

$$\frac{du}{dx} = B_0 + u^2 \tag{27}$$

$B_0(x)$ is a polynomial of even degree, then no polynomial other than

$$u = \pm [\sqrt{-B_0}]$$

can be a solution of (27). If the degree of B_0 is odd, there is no polynomial solution of (27).

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* If A_0, A_1, and A_2 are polynomials, B_1 is usually not a polynomial. There is no intent here to imply that the problems of determining polynomial solutions of equations (23) and (26) are equivalent. That is not true. See Section 15.

Proof: Let the degree of B_0 be even, and write

$$S = [\sqrt{-B_0}];$$

further, define the polynomial Q by the relation

$$-B_0 = S^2 + Q.$$

It is evident that the degree of Q is less than that of S. We may now write (27) in the form

$$\frac{du}{dx} = (u - S)(u + S) - Q. \tag{28}$$

Let u be a polynomial in x. Unless $u = S$ or $u = -S$, the degree of the right-hand member of (28) will be at least as great as the larger of the degrees of u and S. The degrees of u' and Q are each less than the larger of the degrees of u and S. Therefore it follows that the equation cannot be satisfied by any polynomial other than $\pm S$.

Next let the degree of B_0 be odd. Since the degree of u^2 is even and is greater than that of u', and since, in order that the equation be satisfied, the terms in $B_0 + u^2$ of degree greater than that of u' must be identically zero, the degrees of B_0 and u^2 must be the same. This is impossible because the degree of B_0 is odd, while that of u^2 is even. Thus there is no polynomial solution of (27) when B_0 is a polynomial of odd degree. This completes the proof of the theorem.

By the transformation $u = w - \frac{1}{2}B_1$, the equation

$$u' = B_0 + B_1 u + u^2 \tag{26}$$

is transformed into

$$w' = C_0 + w^2,$$

where

$$C_0 = -\frac{1}{4}\left(B_1^2 - 4B_0 - 2\frac{dB_1}{dx}\right).$$

Consequently Theorem 2 is applicable, and we may state

Theorem 3. If in

$$u' = B_0 + B_1 u + u^2 \tag{26}$$

B_0 and B_1 are polynomials in x, and if the degree of

$$\Delta = B_1^2 - 4B_0 - 2\frac{dB_1}{dx}$$

is even, then no polynomial other than

$$u = -\tfrac{1}{2}(B_1 \pm T),$$

where $T = [\sqrt{\Delta}]$, can be a solution of the equation. If the degree of Δ is odd, there is no polynomial solution of (26).

Example. Determine all polynomial solutions of the equation

$$u' = 2x^3 - 3x^2 - 4x + 6 + (x^3 - 2x + 4)u + u^2.$$

Solution. Here we have a particular case of the equation of Theorem 3. We put $B_0 = 2x^3 - 3x^2 - 4x + 6$, $B_1 = x^3 - 2x + 4$, and find with a little computation that

$$\Delta = x^6 - 4x^4 - 10x^2 - 4.$$

Next we obtain $T = [\sqrt{\Delta}]$ by a common square-root-extraction process. Such a computation may be arranged as follows:

$$\begin{array}{rl}
 & x^3 - 2x \cdots \\
 & \sqrt{x^6 - 4x^4 - 10x^2 - 4} \\
 & x^6 \\ \cline{2-2}
\underline{2x^3 - 2x} & \big|\; -4x^4 \\
 & \underline{-4x^4 + 4x^2} \\
\underline{2x^3 - 4x - \frac{7}{x}} & \big|\; -14x^2 \\
 & \underline{-14x^2 \cdots} \\
 & \cdots
\end{array}$$

We thus find that $T = x^3 - 2x$. Next, the two "trial polynomials" as indicated in Theorem 3 are

$$u_1 = -\tfrac{1}{2}(B_1 + T) = -x^3 + 2x - 2 \quad \text{and} \quad u_2 = -\tfrac{1}{2}(B_1 - T) = -2.$$

Of these, it is easily seen by direct substitution that u_1 is a solution, while u_2 is not a solution. We may conclude that

$$u_1 = -x^3 + 2x - 2$$

is the only polynomial solution of the equation of this example.

PROBLEMS

1. Show that $[\sqrt{x^4 - 2x^3 + x - 6}] = x^2 - x - \frac{1}{2}$.

2. Evaluate $[\sqrt{9x^8 - 6x^7 + x^6 + 12x^5 - 10x^4 + x^3 - x + 7}]$.

Ans. $3x^4 - x^3 + 2x - 1$.

3. Evaluate $[\sqrt{x^6 + 8x^4 - x^3 + 7x - 2}]$.

4. Examine for polynomial solutions:

(*a*) $$\frac{du}{dx} = -x^3 + x - 3 - (x - 1)^2 u + u^2.$$

Ans. $u_2 = x^2 - x + 2$.

(*b*) $$\frac{dy}{dx} = -x^4 - 3x^2 + 2 + y^2.$$

Ans. No polynomial solution.

(*c*) $$\frac{du}{dx} = x^3 - (3x - 1)u + u^2.$$

Ans. No polynomial solution.

(*d*) $$\frac{dy}{dx} = 16x^2 - 5 + 8xy + y^2.$$

Ans. $y_1 = -4x - 1,$
$y_2 = -4x + 1.$

5. Examine for polynomial solutions:

(*a*) $$\frac{dy}{dx} = -x^5 - x^2 y + y^2.$$

(*b*) $$\frac{dy}{dx} = 2x^3 - 3x^2 - 2 + (x^3 + 1)y + y^2.$$

(*c*) $$\frac{du}{dx} = x^2 + x - 1 + (2x + 1)u + u^2.$$

(*d*) $$\frac{dw}{dz} = -z^6 - z^5 + z^3 + z^2 w + w^2.$$

6. Solve the equation
$$y' = 1 - x^2 + y^2.$$

7. Solve the equation
$$\frac{dy}{dx} = -1 + xy + y^2.$$

14. Another Theorem on Polynomial Solutions. It is natural to inquire into the conditions under which both trial polynomials would satisfy the equation. By way of answering this question we state and prove the following theorem:

Theorem 4. A necessary and sufficient condition that both trial polynomials of Theorem 3 be solutions of the equation

$$u' = B_0 + B_1 u + u^2 \tag{26}$$

is that Δ *reduce to a constant.*

To prove the necessity we assume that $u_1 = -\frac{1}{2}(B_1 + T)$ and $u_2 = -\frac{1}{2}(B_1 - T)$ both satisfy equation (26). By substituting these into that equation we get $T' = \frac{1}{2}(T^2 - \Delta)$ and $T' = -\frac{1}{2}(T^2 - \Delta)$. From this it follows at once that $T' = 0$ and hence $T = k$, a constant. Since T is constant, so is Δ, and the first part of the theorem is proved.

Now let Δ be constant. Then $T = \sqrt{\Delta}$, $u_1 = -\frac{1}{2}(B_1 + \sqrt{\Delta})$, and

$$B_0 + B_1u_1 + u_1^2 = B_0 - \tfrac{1}{2}B_1^2 - \tfrac{1}{2}B_1\sqrt{\Delta} + \tfrac{1}{4}B_1^2 + \tfrac{1}{2}B_1\sqrt{\Delta} + \tfrac{1}{4}\Delta.$$

But

$$\tfrac{1}{4}\Delta = \tfrac{1}{4}B_1^2 - B_0 - \tfrac{1}{2}B_1',$$

so that

$$B_0 + B_1u_1 + u_1^2 = -\tfrac{1}{2}B_1' = u_1'.$$

In the same manner $u_2 = -\frac{1}{2}(B_1 - \sqrt{\Delta})$ and

$$\begin{aligned} B_0 + B_1u_2 + u_2^2 &= B_0 - \tfrac{1}{2}B_1^2 + \tfrac{1}{2}B_1\sqrt{\Delta} + \tfrac{1}{4}B_1^2 - \tfrac{1}{2}B_1\sqrt{\Delta} + \tfrac{1}{4}\Delta \\ &= -\tfrac{1}{2}B_1' = u_2'. \end{aligned}$$

This completes the proof of Theorem 4.

We proceed in this case to obtain the general solution. Put $u = -\frac{1}{2}(B_1 + k) + v^{-1}$, where $k = \sqrt{\Delta}$. Then $v' - kv + 1 = 0$. Hence, if $k \neq 0$,

$$v = ce^{kx} + k^{-1},$$

where c is an arbitrary constant. Then the general solution of our equation, with Δ constant, is

$$u = -\tfrac{1}{2}(B_1 + k) + (ce^{kx} + k^{-1})^{-1}.$$

It is worth noting that for $c = 0$ and $c \to \infty$ the above solution reduces to the two polynomial solutions.

If $k = 0$, $\Delta = 0$, and the general solution is of the form

$$u = -\tfrac{1}{2}B_1 + \frac{1}{c_1 - x}.$$

15. Trial Solutions. Many forms other than polynomials for B_0 and B_1 suggest possible types of solutions. Often the number of solutions of given type which need to be tried can be greatly restricted by conditions similar to those discussed in this section. It might be interesting to see what results can be found with some other simple forms for B_0 and B_1.

We wish to emphasize that only certain Riccati equations, those of the form (26) with B_0 and B_1 polynomials in x, are affected by the theorems of the preceding two sections. For instance, a Riccati equation, to which the above theorems do not apply, may still have a polynomial solution. Consider, for example, the equation

$$\frac{dy}{dx} = \sin^2 x + \frac{\cos^2 x}{x^2} y^2.$$

It is easy to see that $y = x$ is a solution of this equation.

Naturally we should consider the question of polynomial solutions of the equation

$$Ay' = B_0 + B_1 y + B_2 y^2, \tag{29}$$

where A, B_0, B_1, and B_2 are polynomials in x. One fact which makes such a problem seem radically different from that solved in Section 13 is that the number of possible polynomial solutions is not at all restricted to two, as was true in the special cases treated there.

As an example of an equation with several polynomial solutions, consider the following:

$$\begin{aligned}(9x^4 + 18x^3 + 11x^2 + 2x)\frac{dy}{dx}\\ = -9x^4 - 6x^3 + 4x^2 + (18x^3 + 24x^2 + 4x + 2)y + 3y^2.\end{aligned}$$

This equation has the five polynomial solutions

$$\begin{aligned} y_1 &= x,\\ y_2 &= x^2 + 2x,\\ y_3 &= 3x^3 + 5x^2 + 3x,\\ y_4 &= 2x^2 + \tfrac{7}{3}x,\\ y_5 &= -x^2 - \tfrac{2}{3}x - \tfrac{2}{3}.\end{aligned}$$

See also Problems 4, 5, and 11 at the end of this chapter.

So far as we know, the question of the determination of polynomial solutions of equation (29) has not yet been settled in a satisfactory manner. For the student who cares to attack this problem, we present a few hints. If the equation

$$Ay' = B_0 + B_1 y + B_2 y^2 \tag{29}$$

has a polynomial solution $y = C(x)$, we may simplify the equation by using the substitution $y = u + C$. The equation in u is

$$Au' = B_3u + B_2u^2,$$

where $B_3 = B_1 + 2CB_2$. This equation has $u = 0$ as a polynomial solution and may be written in the convenient form

$$Au' = u(B_3 + B_2u).$$

From this form of the equation for u we may conclude that, if a polynomial solution u has the factor $(x - a)^m$, $m \geqq 1$, then $(x - a)$ is a factor of $A(x)$. This fact immediately restricts the trial polynomials to those whose linear factors are contained in $A(x)$. It is conceivable that further study along these lines will lead to useful restrictions of trial polynomials for equation (29).

16. Application to Differential Operators. Now that we have some knowledge of Riccati equations, we may expect to be able to do a little more with the problem of Section 8, on the resolution of operators. A careful examination of the results of Sections 13, 14, and 8 would show that we are in a position now to discuss more intelligently the factorization of second-order operators of the form

$$\mathrm{D}^2 + a_1\mathrm{D} + a_0,$$

where $\mathrm{D} \equiv \dfrac{d}{dx}$, and a_1 and a_0 are polynomials in x. We shall be concerned only with factorizations of the form

$$\mathrm{D}^2 + a_1\mathrm{D} + a_0 = (\mathrm{D} + b)(\mathrm{D} + c), \tag{30}$$

in which b and c are also polynomials in x.

Now, $(\mathrm{D} + b)(\mathrm{D} + c) = \mathrm{D}^2 + (b + c)\mathrm{D} + bc + c'$, where c' is the derivative of c with respect to x. Hence, if the desired factorization (30) is to exist, we must have

$$b + c = a_1,$$
$$bc + c' = a_0.$$

It is a simple matter to conclude, by eliminating b from those equations, that c must be a polynomial solution of the equation

$$c' = a_0 - a_1c + c^2. \tag{31}$$

We have reduced the above problem to precisely the situation with which Theorem 3, page 38, is concerned. Let us add two definitions to make it easier to express the fact that an operator has a factorization of the type involved in (30). We define a *polynomial differential operator* as an operator which is a polynomial in a variable x and in the operator $\mathrm{D} \equiv \frac{d}{dx}$. We say that a polynomial differential operator B is a *proper divisor* of a polynomial differential operator A if, and only if, conditions (a) to (c) below are satisfied.*

(a) There exists a polynomial differential operator C such that A = BC or such that A = CB.

(b) The degree of B as a polynomial in D is less than the degree of A as a polynomial in D.

(c) The degree of B as a polynomial in D is at least one.

The combination of the above remarks and definitions with the known Theorem 3 gives us all we need to complete the proof of the following theorem.

Theorem 5. The operator $\mathrm{A} = \mathrm{D}^2 + a_1(x)\mathrm{D} + a_0(x)$ *with polynomial coefficients* has (a) *no proper divisors, if the degree of* $\Delta = a_1^2 - 4a_0 + 2a_1'$ *is odd, and* (b) *no proper divisors other than* $\mathrm{B}_1 = \mathrm{D} + \frac{1}{2}a_1 + \frac{1}{2}[\sqrt{\Delta}]$ *or* $\mathrm{B}_2 = \mathrm{D} + \frac{1}{2}a_1 - \frac{1}{2}[\sqrt{\Delta}]$, *if the degree of* Δ *is even.*

It is to be noted particularly that in part (b) of the theorem it is not implied that B_1 or B_2 does divide A. The theorem states merely that no other polynomial differential operator is a proper divisor of A. In practice, if the degree of Δ is even, we need only form the products $\mathrm{B}_1\mathrm{B}_2$ and $\mathrm{B}_2\mathrm{B}_1$ to settle completely the question of factorization in our sense.

In closing this section let us set up the problem of the factorization of the general second-order polynomial differential operator. If we put

$$a_2\mathrm{D}^2 + a_1\mathrm{D} + a_0 = (b_1\mathrm{D} + b_0)(c_1\mathrm{D} + c_0),$$

where all letters except D stand for polynomials in x, we find that the unknown polynomials b_0, b_1, c_0, c_1 must satisfy the equations

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* If we were to discuss differential operators of order higher than two, we would need to modify condition (a) in order to include the possibility that B be an interior divisor of A. That is, we would desire to include the possibility that operators F and G exist such that A = FBG.

$$\begin{aligned} b_1c_1 &= a_2, \\ b_1c_1' + b_1c_0 + b_0c_1 &= a_1, \\ b_1c_0' + b_0c_0 &= a_0. \end{aligned}$$

It is easy to show that (b_1c_0) must be a polynomial solution of the Riccati equation

$$(32) \qquad a_2u' = a_0a_2 + (a_2' - a_1)u + u^2.$$

Again we have reduced the problem of factorization to the determination of polynomial solutions of a Riccati equation, but this time it is not a simple equation of the type we treated successfully in Sections 13 and 14. We are, as was indicated in Section 15, in no position now to go further with this problem.

PROBLEMS

1. Carry through the details in the proof of Theorem 5, page 44.

2. Using the notation of Theorem 5, prove the following result: A necessary and sufficient condition that the operator A have two distinct factorizations in our sense is that Δ be a non-zero constant.

3. Discuss completely the factorization into proper divisors of each of the following differential operators:

(*a*) $\mathrm{D}^2 + \mathrm{D} - x^2 - x - 1$.
(*b*) $\mathrm{D}^2 + x\mathrm{D} + x^3$.
(*c*) $\mathrm{D}^2 + x\mathrm{D} - x + 1$.
(*d*) $\mathrm{D}^2 + (2x - 1)\mathrm{D} + x^2 - x - 1$.

4. With the notation of the last two paragraphs of this section show that $u = b_1c_0$ must satisfy equation (32).

17. The Cross-Ratio Theorem.

A concept which enters projective geometry, the theory of functions, and other branches of mathematics is that of the cross-ratio. The *cross-ratio* of four quantities, y_1, y_2, y_3, and y_4, may be defined as

$$\frac{y_1 - y_3}{y_1 - y_4} \cdot \frac{y_2 - y_4}{y_2 - y_3}.$$

For this ratio we shall use a customary notation, (y_1, y_2, y_3, y_4).

There are twenty-four permutations of four quantities. Hence y_1, y_2, y_3, and y_4 yield twenty-four cross-ratios. We have chosen to refer to

$$(y_1, y_2, y_3, y_4) = \frac{y_1 - y_3}{y_1 - y_4} \cdot \frac{y_2 - y_4}{y_2 - y_3}$$

as the cross-ratio of y_1, y_2, y_3, and y_4. Put

$$(y_1, y_2, y_3, y_4) = \lambda.$$

Then consider another of the twenty-four ratios, say

$$(y_2, y_1, y_3, y_4) = \frac{y_2 - y_3}{y_2 - y_4} \cdot \frac{y_1 - y_4}{y_1 - y_3}.$$

It is easy to see that

$$(y_2, y_1, y_3, y_4) = \frac{1}{\lambda}.$$

Direct computation will show that the twenty-four cross-ratios of four quantities yield only six distinct values, which in our notation are

$$\lambda, \quad \frac{1}{\lambda}, \quad 1 - \lambda, \quad \frac{1}{1 - \lambda}, \quad \frac{\lambda}{\lambda - 1}, \quad \frac{\lambda - 1}{\lambda}.$$

These are the elements of the cross-ratio group.

It is beyond the scope of this book to consider the many interesting phenomena related to this group, even when they are connected with our subject material. We shall encounter one such connection at the end of Section 49. For material on the cross-ratio group see nearly any elementary textbook on group theory or on projective geometry.

Our interest here is centered in the following result:

Theorem 6. If y_1, y_2, y_3, *and* y_4 *are solutions of the Riccati equation*

$$\frac{dy}{dx} = A_0(x) + A_1(x)y + A_2(x)y^2, \tag{23}$$

the cross-ratio (y_1, y_2, y_3, y_4) *is constant.*

Let v_1 and v_2 be linearly independent solutions of the equation $v'' + Pv' + Qv = 0$. Let v_3 and v_4 be solutions of the same equation. Then constants a_1, a_2, b_1, and b_2 exist such that

$$v_3 = a_1v_1 + a_2v_2,$$

$$v_4 = b_1v_1 + b_2v_2.$$

Hence*

$$\begin{vmatrix} v_3 & v_4 \\ v_3' & v_4' \end{vmatrix} = \begin{vmatrix} a_1v_1 + a_2v_2 & b_1v_1 + b_2v_2 \\ a_1v_1' + a_2v_2' & b_1v_1' + b_2v_2' \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix}.$$

We have proved the following

Lemma. If v_1, v_2, v_3, *and* v_4 *are solutions of* $v'' + Pv' + Qv = 0$ *such that* $v_1v_2' - v_1'v_2 \neq 0$, *then*

$$\frac{v_3v_4' - v_3'v_4}{v_1v_2' - v_1'v_2}$$

is constant.

Consider now the cross-ratio (y_1, y_2, y_3, y_4) of Theorem 6. If we change the dependent variable in the Riccati equation

$$\frac{dy}{dx} = A_0(x) + A_1(x)y + A_2(x)y^2 \tag{23}$$

by

$$y = -\frac{u'}{A_2u},$$

then, as we know, that equation is transformed into a linear equation of the second order. Let u_i be any of the functions determined, except for a constant multiplier, by $y_i : i = 1, 2, 3, 4$ by means of the above transformation. Then

$$(y_1, y_2, y_3, y_4) = \frac{y_1 - y_3}{y_1 - y_4} \cdot \frac{y_2 - y_4}{y_2 - y_3} = \frac{\dfrac{u_1'}{u_1} - \dfrac{u_3'}{u_3}}{\dfrac{u_1'}{u_1} - \dfrac{u_4'}{u_4}} \cdot \frac{\dfrac{u_2'}{u_2} - \dfrac{u_4'}{u_4}}{\dfrac{u_2'}{u_2} - \dfrac{u_3'}{u_3}}$$

$$= \frac{u_3u_1' - u_3'u_1}{u_4u_1' - u_4'u_1} \cdot \frac{u_4u_2' - u_4'u_2}{u_3u_2' - u_3'u_2}.$$

But no two of the u_i can be linearly dependent since the corresponding y_i are distinct. Hence the u_i satisfy the conditions of the lemma, and Theorem 6 follows at once.

One immediate application of Theorem 6 is that, if y_2, y_3, and y_4

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[*] The determinant $(v_1v_2' - v_1'v_2)$ is called the Wronskian of the functions v_1 and v_2. The identical vanishing of the Wronskian of two functions is easily seen to be a necessary and sufficient condition that the two functions be linearly dependent. The Wronskian is named for H. Wronski (1778–1853), a Polish mathematician.

are known particular integrals of a Riccati equation, then the general solution y is given by

$$\frac{y - y_3}{y - y_4} \cdot \frac{y_2 - y_4}{y_2 - y_3} = c,$$

where c is an arbitrary constant.

We know how to obtain the general solution of a Riccati equation by means of two integrations when one particular solution is known. We know how to obtain that general solution without integrations when three particular solutions are known. As might be expected, we can easily show how to determine the general solution with one integration when two particular solutions are known. The details are left to the student.

Before proceeding to sketch two other proofs of Theorem 6, we derive a well-known formula which could have been used to prove the lemma.

Let v_1 and v_2 be solutions of $v'' + Pv' + Qv = 0$. Then

$$v_1v_2'' + Pv_1v_2' + Qv_1v_2 = 0,$$

$$v_2v_1'' + Pv_2v_1' + Qv_2v_1 = 0,$$

so that

$$v_1v_2'' - v_1''v_2 + P(v_1v_2' - v_1'v_2) = 0.$$

Now $(v_1v_2' - v_1'v_2)' = v_1v_2'' - v_1''v_2$. If $v_1v_2' - v_1'v_2 \not\equiv 0$, we may write

$$\frac{(v_1v_2' - v_1'v_2)'}{v_1v_2' - v_1'v_2} + P = 0. \tag{33}$$

From equation (33) we have at once

$$v_1v_2' - v_1'v_2 = ce^{-\int P\,dx} \tag{34}$$

where c is an arbitrary constant. Equation (34) is called Abel's formula.*

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* Niels Henrick Abel (1802–1829) was a Norwegian mathematician. During his short, tragic life he made many important contributions to mathematics.

PROBLEMS

1. In Problem 1, Section 12, we found that the equation

$$y' = x^2 - 2 + 2xy + y^2$$

has the particular solution $y = 1 - x$. Show that $y = -x - \coth x$ is also a solution of this equation. Determine another polynomial solution of the equation, noting the application of Theorem 4, Section 14. Finally, write down the general solution of this equation by using all three indicated particular solutions.

2. In Section 15 it was pointed out that the equation

$$(9x^4 + 18x^3 + 11x^2 + 2x)\frac{dy}{dx} = -9x^4 - 6x^3 + 4x^2 + (18x^3 + 24x^2 + 4x + 2)y + 3y^2$$

has the five polynomial solutions

$$\begin{aligned} y_1 &= x, \\ y_2 &= x^2 + 2x, \\ y_3 &= 3x^3 + 5x^2 + 3x, \\ y_4 &= 2x^2 + \tfrac{7}{3}x, \\ y_5 &= -x^2 - \tfrac{2}{3}x - \tfrac{2}{3}. \end{aligned}$$

Show that $(y_1, y_2, y_3, y_4) = \frac{1}{2}$. Obtain the constant value of the cross-ratio of some other combination of these particular solutions.

3. Complete the solution of the equation of Problem 2.

18. Other Proofs of Theorem 6. We now consider very briefly two other proofs of Theorem 6. First, we know that the general solution of

$$\frac{dy}{dx} = A_0(x) + A_1(x)y + A_2(x)y^2 \tag{23}$$

may be written in the form

$$y = \frac{bf_1(x) + f_2(x)}{bf_3(x) + f_4(x)}. \tag{25}$$

Each of our particular solutions corresponds to some value of the arbitrary constant b in (25). Write

$$y_i = \frac{b_if_1(x) + f_2(x)}{b_if_3(x) + f_4(x)},$$

for $i = 1, 2, 3, 4$. Then we may prove Theorem 6 by constructing (y_1, y_2, y_3, y_4) and showing by algebraic simplification that the cross-ratio reduces to a function of the constants b_i and is independent of the functions $f_i(x)$. It is necessary in the proof to assume that the determinant of the coefficients in (25) does not vanish. That is, $f_1f_4 - f_2f_3 \neq 0$. It is a simple matter to show that this assumption induces no loss of generality; indeed, it must be made or (25) is not the general solution in question. It would contain no arbitrary constant. Finally, the constant value of (y_1, y_2, y_3, y_4) is (b_1, b_2, b_3, b_4).

Another proof makes use of the fact that from the four equations

$$y_i' = A_0 + A_1y_i + A_2y_i^2,$$

$i = 1, 2, 3, 4$, we may conclude that

$$\begin{vmatrix} y_1' & 1 & y_1 & y_1^2 \\ y_2' & 1 & y_2 & y_2^2 \\ y_3' & 1 & y_3 & y_3^2 \\ y_4' & 1 & y_4 & y_4^2 \end{vmatrix} = 0.$$

By elementary methods we may, with the assumption that the y_i are distinct, reduce the above equation to the form

$$\frac{d}{dx}(y_1, y_2, y_3, y_4) = 0.$$

Hence Theorem 6 follows.

MISCELLANEOUS PROBLEMS

1. Obtain the general solution of

$$\frac{dy}{dx} = 1 + y^2$$

by purely elementary means (separation of variables) and show, by doing so, that this solution may be put into the form (25), page 49. Solve this problem again by obtaining one or more inspection solutions and applying the pertinent parts of the theory of Riccati equations. There are at least four very simple particular solutions.

2. Solve

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1 - 2x^2)y = 0$$

by introducing a Riccati equation. See page 32.

3. Solve the equation of Problem 2 by some other device.

4. Show that 1, x, and x^2 are particular solutions of

$$x(x^2 - 1)y' + x^2 - (x^2 - 1)y - y^2 = 0,$$

and hence obtain the general solution of this equation.

5. Obtain another polynomial solution of the equation in Problem 4. Next find the cross-ratio of the four polynomial solutions of the equation of Problem 4.

6. Solve the equation

$$xu' = 4x^3 - (4x^2 - 1)u + xu^2$$

by first obtaining a solution which is linear in x.

7. Verify that the differential operator

$$(x^2 - x^3)\mathrm{D}^2 + (2x^2 - 4x)\mathrm{D} - 2x + 6$$

has the three distinct factorizations

(*a*) $(x\mathrm{D} - 2)[(x - x^2)\mathrm{D} + 2x - 3]$.

(*b*) $(x\mathrm{D} - 3)[(x - x^2)\mathrm{D} + x - 2]$.

(*c*) $[(x - x^2)\mathrm{D} + x - 3](x\mathrm{D} - 2)$.

8. Use the particular solution $y = x$ to complete the solution of the equation

$$(x^3 - x^2 + x + 2)y' = x^2 + 2 - (x^2 + 2x - 1)y + 2xy^2.$$

9. Prove that: If A_0, A_1, and A_2 are polynomials in x, then the equation

$$y' = A_0 + A_1y + A_2y^2$$

cannot have three polynomial solutions. *Hint:* Assume that the equation has at least the one polynomial solution $y = P(x)$. Then introduce a new dependent variable $u = y - P(x)$.

10. Show that, if

$$y = \frac{cf_1(x) + f_2(x)}{cf_3(x) + f_4(x)},$$

where c is an arbitrary constant, and $f_1f_4 - f_2f_3 \not\equiv 0$, then y must satisfy, and is the general solution of, the Riccati equation

$$\begin{vmatrix} f_1 & f_2 \\ f_3 & f_4 \end{vmatrix} y' = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} + \left\{ \begin{vmatrix} f_2 & f_3 \\ f_2' & f_3' \end{vmatrix} + \begin{vmatrix} f_4 & f_1 \\ f_4' & f_1' \end{vmatrix} \right\} y + \begin{vmatrix} f_3 & f_4 \\ f_3' & f_4' \end{vmatrix} y^2.$$

11. Use the result of Problem 10 to construct a Riccati equation with polynomial coefficients and any desired number of polynomial solutions in the manner described below. Choose $f_1(x) \equiv 0$, and choose any desired polynomials as $f_3(x)$ and $f_4(x)$. Let $c_1, c_2, \cdots, c_n$ be specific values of the arbitrary constant c. Then choose

$$f_2 = (c_1f_3 + f_4)(c_2f_3 + f_4) \cdots (c_nf_3 + f_4).$$

The resultant Riccati equation will have the $(n + 1)$ polynomial solutions $y = 0$ and $y_i = \dfrac{f_2}{c_if_3 + f_4}$; $i = 1, 2, \cdots, n$. If you wish to make the result less obvious, choose a polynomial $P(x)$ and introduce a new dependent variable $u = y + P(x)$. The new equation will still have $(n + 1)$ polynomial solutions, but $u = 0$ may not be among them. Carry through the details in one or more numerical cases with simple polynomials for f_3 and f_4. In particular, use $f_3 = 1$, $f_4 = x$, $c_1 = 0$, $c_2 = 1$, $c_3 = -1$, $c_4 = 2$, $c_5 = \frac{1}{2}$, $P(x) = 2x$.

Ans. The equation

$$\begin{aligned}(2x^5 + 5x^4 - 5x^2 - 2x)u' = {} & -16x^5 - 30x^4 + 2x^2 \\ & + (10x^4 + 20x^3 - 2x - 2)u - 2u^2\end{aligned}$$

has the six polynomial solutions

$$\begin{aligned}u_1 &= 2x, \\ u_2 &= x^4 + \tfrac{5}{2}x^3 - \tfrac{1}{2}x - 1, \\ u_3 &= x^4 + \tfrac{3}{2}x^3 - \tfrac{3}{2}x^2 + x, \\ u_4 &= x^4 + \tfrac{7}{2}x^3 + \tfrac{7}{2}x^2 + 3x, \\ u_5 &= x^4 + \tfrac{1}{2}x^3 - x^2 + \tfrac{3}{2}x, \\ u_6 &= x^4 + 2x^3 - x^2.\end{aligned}$$

CHAPTER III

PRELIMINARY MATERIAL ON COMPLEX VARIABLES

19. Remarks. In the attempt to solve linear differential equations with the aid of power series we soon encounter the need for a knowledge of the elements of the theory of functions of a complex variable. We shall now consider briefly certain fundamental ideas and facts from that theory. In view of the fact that many semester courses can be devoted to complex variables without exhausting the wealth of material already known in that field, we must, of course, avoid even the mention of certain vast branches of that subject. We shall confine ourselves almost entirely to a descriptive account of those elementary portions of complex variables the need for which appears in the remainder of our present study of linear differential equations. We are indebted to Professor C. A. Hutchinson of the University of Colorado for the organization and choice of much of the material in this chapter. We follow quite closely certain of his lecture notes of 1932–1933.

20. Single-Valued Analytic Functions. Consider a complex variable $z = x + iy$, where x and y are independent real variables and $i = \sqrt{-1}$. We often represent z as a point in a complex z plane (or xy plane) with a rectangular xy coordinate system. By an *open region* in the z plane we mean the set of points interior to a " simple closed curve " (most often a circle will be used) excluding the boundary points, those on the closed curve. By a *closed region* we mean an open region plus all its boundary points.

We say that $w = u + iv$ is a *single-valued function* of z when u and v are real single-valued functions of x and y. At this stage w is merely a composite of two independent real functions of two real variables. We shall be concerned, however, with only a particularly interesting subset of these functions, the analytic ones (to be defined soon).

Let x and y receive independent increments Δx and Δy. Then z receives an increment $\Delta z = \Delta x + i\,\Delta y$. If w is a single-valued function of z, w receives an increment $\Delta w = \Delta u + i\,\Delta v$. We define the derivative of w with respect to z as

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z},$$

where the limit, if it is to exist, must have a single value independently of the manner in which Δz approaches zero. It can be shown that, if $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ are continuous at a point, then a necessary and sufficient condition for the existence of this derivative at that point is the satisfaction by u and v of the famous Cauchy-Riemann* equations,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}, \end{aligned} \tag{35}$$

throughout some neighborhood of the point.

We call a function $w(z)$ *holomorphic* or *analytic* at a point $z = z_0$ if, and only if, the above derivative $\frac{dw}{dz}$ exists at each point in some neighborhood of that point. If a function is holomorphic at every point of a region, it is said to be analytic in that region. An *analytic function of a complex variable* is one which is analytic at each point of some region. Alternative terminologies, " uniform " and " regular," we shall avoid. The term regular has a distinct meaning in our particular field.

Let us now inspect a few examples.

EXAMPLE 1. $w = x - iy$. Here $\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = -1$, so that w is not an analytic function of z in any region in the finite plane.

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[*] Augustin-Louis Cauchy (1789–1857) was a French mathematician. In addition to producing a great volume of other mathematical research, he may be said to have founded the modern theory of functions of a complex variable.

G. F. B. Riemann (1826–1866) was a German mathematician. His contributions to the theory of functions of a complex variable have had a profound influence. We shall see some of his other work in later chapters.

A large number of the mathematicians of the last two centuries engaged themselves to some extent in the study of the subjects we are considering in this course.

EXAMPLE 2. $w = x^2 - y^2 + 2ixy = z^2$. Here $\dfrac{\partial u}{\partial x} = 2x = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -2y = -\dfrac{\partial v}{\partial x}$. Hence w is an analytic function holomorphic at every point of the finite plane.

EXAMPLE 3.

$$w = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}.$$

Here

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \qquad \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}, \qquad \frac{\partial v}{\partial x} = +\frac{2xy}{(x^2 + y^2)^2},$$

and the Cauchy-Riemann equations are satisfied in every region which does not include the point $z = 0$. At this point $x = y = 0$, and the various partial derivatives do not exist.

A point at which the function w is holomorphic is called an ordinary point of that function. The common alternative terminology " regular point " is unwise for us.

PROBLEMS

Determine directly from u and v the regions in which the Cauchy-Riemann equations are satisfied for each of the following functions of the complex variable $z = x + iy$.

1. $w = u + iv = x^2 + y^2 = |z|^2$.

Ans. Not in any region.

2. $w = e^z = e^{x+iy} = e^x (\cos y + i \sin y)$.

Ans. In every finite region.

3. $w = \sin z = \sin x \cosh y + i \cos x \sinh y$.

4. $w = \dfrac{1}{z - 1}$.

Ans. In every finite region not including the point $z = 1$.

21. Singularities. Any point at which the derivative fails to exist we call a *singularity* of the corresponding function. We still restrict our remarks to single-valued analytic functions. For these functions singularities are divided into two main classes, poles and essential singularities. We now define a pole.

DEFINITION 2. The point z_0 is called a *pole* of the single-valued analytic function w of the complex variable z if, and only if, there exists an integer $k > 0$ such that

$$(z - z_0)^k w(z)$$

is analytic, and not zero, at $z = z_0$.

The number k is called the order of the pole at z_0. A pole of order one is usually called a *simple pole.* Note that the function in Example 3, page 55, has a simple pole at $z = 0$.

For a single-valued function any singularity which is not a pole is called an *essential singularity.* If z_0 is an essential singularity such that it is possible to surround z_0 with a circle of positive radius which encloses no other essential singularity, then z_0 is called an *isolated essential singularity.* Next to poles these are the simplest singularities. We shall sometimes refer to a *completely isolated singularity,* by which we mean a singularity isolated from all other singularities. Such a singularity may be either a pole or an essential singularity isolated even from poles.

One distinction between poles and essential singularities of single-valued functions is the following: if $f(z)$ has a pole at z_0, then $\dfrac{1}{f(z)}$ is holomorphic at z_0 (indeed, it has a zero at z_0); if $f(z)$ has an essential singularity at z_0, so does $\dfrac{1}{f(z)}$. In a sense $f(z)$ is " properly infinite " at a pole. There are, of course, other more striking distinctions which would be discussed in a course in the theory of functions.

We quote Professor Hutchinson directly in the next paragraph.

> It becomes necessary to consider the concept of infinity in connection with complex variables. If the absolute value of z, $|z| = r = \sqrt{x^2 + y^2}$, becomes large without limit, we say that z becomes infinite, $z \longrightarrow \infty$. Consider the relation $w = \dfrac{1}{z}$. To every point in the z plane, with one exception, there corresponds exactly one point in the w plane, and conversely, again with one exception. The exceptions are $z = 0$ and $w = 0$, to which there are no corresponding points. To remove this exception, we " complete " the complex planes by introducing fictitious, or ideal, " points " $z = \infty$ and $w = \infty$. The justification for considering $z = \infty$ as a point may be made very vivid by using a different geometrical representation of complex numbers. Let a sphere of diameter unity be tangent

to the z plane at the origin. Call the point of tangency the south pole of the sphere. Project the points of the sphere onto the plane stereographically by lines radiating from the north pole. Then there is a one-to-one correspondence of the points of the plane with those of the sphere [excepting only the north pole]. As a point on the sphere approaches the north pole in any manner whatever, the corresponding point in the z plane moves away from the origin, its distance increasing without limit. We therefore mark the north pole $z = \infty$. We now have a complete one-to-one correspondence between the points of the sphere and those of the completed z plane. This representation of complex numbers is called the "Neumann sphere."

We next state Liouville's theorem.* *If $f(z)$ has no singularity, z finite or infinite, $f(z)$ is a constant.*

A polynomial in z is holomorphic at every finite z and has a pole at $z = \infty$, the order of the pole being the degree of the polynomial.

The function

$$f(z) = \frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}; \qquad a_0 \neq 0, \quad b_0 \neq 0,$$

the ratio of two polynomials, assumed to have no variable factors in common, is holomorphic at every finite z except for the zeros of the denominator, at which zeros $f(z)$ has poles. This $f(z)$ is holomorphic at $z = \infty$, if $m \geqq n$. If $m < n$, $z = \infty$ is a pole of order $(n - m)$ for $f(z)$.

We shall use in later chapters the result: *If $f(z)$ has no singularities other than poles in the finite plane or at infinity, then $f(z)$ is the ratio of two polynomials in z.*

The function $e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$ has only one singularity, an essential singularity at $z = \infty$. The functions $\sin z$ and $\cos z$ are also holomorphic at every point except $z = \infty$ where each has an essential singularity. On the other hand, $\csc z$ has simple poles at $z = n\pi$ for each integral n, positive, negative, or zero, and it has an essential singularity at $z = \infty$. These are all the singularities of $\csc z$. Log z is not a single-valued function.

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* Joseph Liouville (1809–1882) was a French mathematician who contributed interesting and important results in many branches of mathematics. With regard to the name "Liouville's theorem" see a footnote in Whittaker and Watson, 4th ed., 105.

PROBLEMS

1. Locate and classify the singularities of the following functions:

(a) $4z^3 + 2z - 1$.

(b) $\dfrac{z+2}{z^2-3}$.

(c) $\dfrac{4z^3+i}{z^3-1}$.

(d) $\dfrac{5z^3-2z+1}{z^2+4}$.

(e) $\sec z$.

2. Determine the order of each of the poles of the following:

(a) $\dfrac{z+2}{z^2(z-1)(z^2+4)^3}$.

(b) $z \csc z$.

(c) $\tan z$.

(d) $z \csc^2 z$.

3. Determine the general form of a function which is analytic everywhere except for simple poles at $z = z_1$, $z = z_2$, and $z = z_3$.

Ans. $F(z) = \dfrac{Az^2 + Bz + C}{(z-z_1)(z-z_2)(z-z_3)}$, or

$$F(z) = \frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \frac{A_3}{z-z_3}.$$

22. Power Series in Complex Variables. If $f(z)$ has $z = z_0$ as an ordinary point, then $f(z)$ may be expanded in a unique power series " about z_0,"

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

which represents (is convergent to the values of) $f(z)$ inside the circle $|z - z_0| = R$, passing through the singularity of $f(z)$ nearest to $z = z_0$. Further, the above series is divergent for z outside the circle, i.e., for $|z - z_0| > R$. The determination of the convergence behavior on the circle itself is a difficult problem which does not concern us here. The number R is called the radius of convergence of the power series. If $f(z)$ has no singularities in the finite plane, its power series is convergent within a circle of any radius, no matter how large. The function is then called an integral, or entire, function. Note also that in the power series expansion $a_n = \frac{f^{(n)}(z_0)}{n!}$.

Suppose that z_0 is a singularity of $f(z)$. Then no such expansion as the above is valid, but, if z_0 is a completely isolated singularity, that is, if it is a pole or an essential singularity isolated from both poles and other essential singularities, then $f(z)$ may be expanded in the Laurent series,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.$$

The region of convergence of a Laurent series is an annular region bounded by the two circles $|z - z_0| = R_1$ and $|z - z_0| = R_2$, where $0 < R_1 < R_2$, R_1 may be made as small as desired, and R_2 is such that the larger circle passes through the singularity of $f(z)$ nearest to, but distinct from, z_0. If the point z_0 is a pole of order k for the function $f(z)$, the term $a_{-k}(z - z_0)^{-k}$ will actually be present in the above expansion and we shall have $a_n = 0$ for all $n < -k$. If z_0 is an essential singularity isolated from all other singularities, then actually an infinity of negative powers of $(z - z_0)$ is present in the Laurent series.

The statements in the above paragraph must not be construed to mean more than they say. The series

$$\sum_{n=-\infty}^{\infty} b_n(z - z_0)^n$$

is called a two-way power series. It is, for example, not true that, if an infinity of negative powers is present, a two-way power series necessarily represents a function with an essential singularity at $z = z_0$. It may represent an analytic function which is even holomorphic at $z = z_0$. For example, the expansion

$$\frac{1}{z-1} = \frac{1}{z}\,\frac{1}{1 - \frac{1}{z}} = \sum_{n=0}^{\infty} z^{-n-1} = \sum_{n=1}^{\infty} z^{-n} = \sum_{n=-\infty}^{-1} z^n$$

is valid for $|z| > 1$. Also

$$\frac{2}{2-z} = \frac{1}{1 - \frac{z}{2}} = \sum_{n=0}^{\infty} 2^{-n} z^n$$

is valid for $|z| < 2$. Hence in the annular region $1 < |z| < 2$ we may write

$$\frac{1}{z-1} + \frac{2}{2-z} = \frac{z}{(z-1)(2-z)} = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where, for $n \geqq 0$, $c_n = 2^{-n}$, and for $n < 0$, $c_n = 1$. Here each power of z is actually present, yet the series represents (in the annular region $1 < |z| < 2$) a function which is holomorphic at $z = 0$.

Here we have an expression, the above two-way series, which may be taken as the definition of a certain analytic function in the annular region $1 < |z| < 2$. Yet we have spoken of that function having a singularity at $z = 0$, a point outside the domain of definition. This is possible because, if two analytic functions have identical values throughout some region, then, when their domains of definition are extended by the process described in the next section, the two functions are identical wherever either is defined. Hence we speak of them as the same function.

Restricting ourselves to single-valued functions with only poles or isolated essential singularities, we may describe the general situation with regard to two-way series about $z = z_0$ as follows. Mark the singularities of $f(z)$ in the z plane. Through each singular point in the finite plane construct a circle with center at z_0. Some or all of these circles may coincide. If z_0 is a singularity, isolated from all other singularities, of $f(z)$, construct an additional circle of arbitrarily small radius $R_0 > 0$ and with center at z_0. This circle replaces a circle of zero radius which might have been included in the set of circles first constructed. If $z = \infty$ is a singularity, isolated from all other singularities, of $f(z)$, or if $z = \infty$ is an ordinary point of $f(z)$, construct another circle with center at z_0 and of arbitrarily large radius R_∞, at least exceeding any other radius involved.

If $f(z)$ is analytic in an annular region bounded by two circles of this set, then $f(z)$ may be expanded into a two-way series which is absolutely and uniformly convergent in any closed region lying entirely within the annular region.

This is Laurent's theorem.* There are special cases of particular interest, namely the ordinary power series when $f(z)$ is holomorphic at z_0 and the Laurent series when $f(z)$ has a completely isolated singularity at z_0.

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* See Whittaker and Watson, *op. cit.*, 100.

It is definitely worth while for the student to take an example such as

$$A(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{z^2+1},$$

or

$$B(z) = A(z) + \frac{1}{z^3},$$

and work out in detail the various expansions possible in two-way series about $z = 0$. Suppose that we wish to expand $A(z)$ in a series valid in $2 < |z| < 3$. We may expand $\frac{1}{z-3}$ in a series of non-negative integral powers of z valid in $|z| < 3$. We may expand $\frac{1}{z-2} + \frac{1}{z^2+1}$ in a series of non-positive integral powers of z valid in $|z| > 2$. The sum of these two series is the desired expansion of $A(z)$.

We include one more example. The function $\csc \frac{1}{z}$ has simple poles at $z = \infty$ and at $z = \frac{1}{n\pi}$ for each positive and each negative integer n, and it has an essential singularity at $z = 0$. These are the only singularities of the function. The singularity at $z = 0$ is an isolated essential singularity, but it is not isolated from the poles of the function. There is an interesting expansion valid in $\frac{1}{\pi} < |z| < R_\infty$, where R_∞ may be arbitrarily large. It is

$$\csc \frac{1}{z} = z + \sum_{n=0}^{\infty} \frac{2(2^{2n+1}-1)}{(2n+2)!} B_{n+1} z^{-2n-1},$$

where the numbers B_{n+1} are called the Bernoulli numbers* and may be defined by

$$\frac{(2^{2n+1}-1)\pi^{2n+2}}{(2n+2)!} B_{n+1} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{2(n+1)}},$$

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* Named after Jakob (James) Bernoulli (1654–1705), one of the best mathematicians in the mathematically famous Bernoulli family. See Cajori's *History of Mathematics*, revised ed., 220–222.

for $n \geqq 0$. The Bernoulli numbers are rational numbers and have many interesting properties. They are met in certain topics in the theory of numbers as well as in the study of infinite series.

We shall close this section with two remarks on ordinary power series. First, *a necessary and sufficient condition that a power series vanish identically, that is, vanish for each point in some region entirely within its circle of convergence, is that each of its coefficients be zero.*

Next, if the power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

has a radius of convergence $R > 0$, then for any number r such that $0 < r < R$ there exists a constant M such that

$$|a_n| < \frac{M}{r^n}$$

for all positive integral n. The above inequality plays a frequent role in the proofs of classical analysis and will be important in two proofs in the next chapter.

23. Analytic Continuation. Within its circle of convergence, $|z - z_0| < R$, the series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{36}$$

may be used to define an analytic function $f(z)$, the sum of the series at each point. It is natural to ask whether this function has meaning, more strictly whether it is analytic, at any points outside the circle of convergence. If so, can we obtain the values of $f(z)$ in some region lying within $|z - z_0| > R$ from the series (36)? If $f(z)$ is holomorphic at some point on the circle of convergence of (36), then the answer to both these questions is in the affirmative. A process by which we obtain $f(z)$ for points outside the original region of definition is called *analytic continuation.* We sketch one, probably the oldest, such process. Many writers in using the term " analytic continuation " have reference solely to this particular process.

Suppose $f(z)$ to be defined in $|z - z_0| < R$ by the series (36). Then we may choose a point z_1 within the circle of convergence and compute from the series the values of the various derivatives

$f^{(n)}(z_1)$; $n = 0, 1, 2, \cdots$, of the function $f(z)$ at the point $z = z_1$. Then by Taylor's theorem we have

$$f(z) = \sum_{n=0}^{\infty} b_n(z - z_1)^n,$$

where $b_n = \dfrac{f^{(n)}(z_1)}{n!}$. This power series will converge in a circle extending from z_1 as a center to the singularity of $f(z)$ nearest z_1. In justification of this process recall that a power series is unlimitedly differentiable in any closed region lying entirely within its circle of convergence. Unless we were able to "recognize" the function from the series (36), we should not know the location of this nearest singularity. It may not, of course, even be on the original circle of convergence.

The process outlined above may be repeated indefinitely. The following exercise is instructive. Mark in the z plane a few random points as singularities of some function. Then proceed gradually to enlarge the domain of definition of the function by this process of continuation, noting the circles of convergence as you continue — without performing the actual computations. It appears that this process is not practical from the standpoint of direct computation.

One more point may well be noted here. There is no guarantee that the series about z_1 will have its circle of convergence extending beyond the original circle of convergence; it may not. Many times such a result means that a poor choice was made in taking z_1. Indeed, in such a case, z_1 was chosen on a line joining z_0 with a singular point on the original circle of convergence. It may be that no matter where z_1 is chosen we do not extend the region of definition of $f(z)$ beyond the original one. Then there is no ordinary point on the circle of convergence and that circle is called a *natural boundary* for the function. As an example,

$$\sum_{n=0}^{\infty} z^{n^2}$$

has the circle $|z| = 1$ as a natural boundary.

24. Conformal Mapping. Suppose that we have a relation between the complex variables $w = u + iv$ and $z = x + iy$, where x, y, u, and v are real. If w is a single-valued function of z, we may associate with each point, excepting essential singularities of $w(z)$, in the z plane its corresponding point $w(z)$ in the w plane. If z_0 is a pole of $w(z)$, then to z_0 corresponds $w = \infty$. We study the manner in which points, curves, regions, etc., of the two planes correspond. We say that we are "mapping" the z plane onto the w plane by means of the function or transformation $w(z)$.

We shall consider a specific example, $w = \frac{1}{z}$. See Example 3, page 55. We have at once $u = \frac{x}{x^2 + y^2}$, $v = \frac{-y}{x^2 + y^2}$ and $x = \frac{u}{u^2 + v^2}$, $y = \frac{-v}{u^2 + v^2}$. For some purposes, such as the study of this particular transformation $w = \frac{1}{z}$, we find the introduction of polar coordinates in the complex plane to be helpful. Let us put $z = re^{i\theta}$, where $r = |z| = \sqrt{x^2 + y^2}$ is called the *modulus* of z and $\theta = \arctan \frac{y}{x}$ is called the *argument* of z. With a similar representation $w = \rho e^{i\varphi}$, we find that $w = \frac{1}{z}$ becomes $\rho = \frac{1}{r}$ and $\varphi = -\theta$.

For a geometric interpretation of the above transformation we use a convention frequently met in complex variables. We shall plot both z and w in the same plane, with the same coordinate axes. We shall speak of the transformation as mapping the plane onto itself and shall say that a point has been transformed into another, or the same, point in the plane. Now with the aid of the polar coordinate representation we can see that $w = \frac{1}{z}$ amounts to an inversion with respect to the unit circle and a reflection with respect to the axis of reals (x axis). That is, we join the point z and the origin with a straight line. Then we determine the point which is on this line in the same direction from the origin as z is but with its distance from the origin the reciprocal of the distance of z from the origin. This is inversion of z with respect to the unit circle. The point w is symmetric to the point described above with respect to the x axis. The geometric description takes care of the transformation of all z points except $z = 0$ and $z = \infty$. But we already know that these two points are interchanged by $w = \frac{1}{z}$.

It is worth while to study in detail the mapping of particular points, curves, and regions by $w = \frac{1}{z}$. Note, for instance, that circles, including straight lines as limiting cases, always transform into circles, or straight lines. Note that the hyperbola $x^2 - y^2 = 1$ transforms into the lemniscate $u^2 - v^2 = (u^2 + v^2)^2$, an example of the fact that the degree of a curve is not invariant under this transformation.

The above transformation is a special case of

$$w = \frac{az + b}{cz + d},$$

the general bilinear or linear fractional transformation. This transformation will prove useful to us later, particularly in the work on the hypergeometric function. A detailed discussion of the properties of the linear fractional transformation may be found in such books as Townsend's *Functions of a Complex Variable.* We mention only two properties here. The transformation may be performed geometrically as a finite sequence of inversions with respect to circles and reflections with respect to straight lines. Finally note that there are essentially three constants in the transformation. Because of this an important property follows at once. Any three points z_1, z_2, and z_3 may be transformed by a linear fractional transformation to any three points w_1, w_2, and w_3. We shall make much use of the special case which transforms the points z_1, z_2, and z_3 into the points 0, 1, and ∞, namely,

$$w = \frac{z_2 - z_3}{z_2 - z_1} \cdot \frac{z - z_1}{z - z_3}.$$

PROBLEMS

1. Use the transformation $w = \frac{1}{z}$ to map each of the following from the z plane onto the w plane.

(*a*) The points $z = 2 + i$, $z = i$, $z = -1 - i$, $z = 1$, $z = \frac{1}{2} + \frac{1}{2}i$.
(*b*) The line $x + y = 1$.

(*c*) The line $y = 2x$.
(*d*) The circle $|z| = 3$, i.e., $x^2 + y^2 = 9$.
(*e*) The circle $|z| = \frac{1}{2}$.
(*f*) The circle $|z - 1| = 1$.
(*g*) The parabola $y^2 = 4(x - 1)$.

2. Determine the linear fractional transformation which maps the points $z = 0$, $z = -1$, and $z = 1$, respectively, onto the points $w = 0$, $w = 2$, and $w = \infty$. Find all the points which are invariant under this transformation, that is, points $z = z_1$ such that $w = z_1$ also. Note that one of these points, $z = 0$, has been given.

3. Determine the linear fractional transformation which maps the points $z = 1$, $z = 4$, and $z = -1$, respectively, onto the points $w = 0$, $w = \frac{1}{2}$, and $w = -2$. Find all the invariant points for this transformation.

4. Determine the most general linear fractional transformation which leaves z_1 and z_2 invariant.

Ans. $$w = \frac{(d + z_1 + z_2)z - z_1 z_2}{z + d}.$$

CHAPTER IV

SOLUTION IN SERIES

25. An Introductory Example. In the previous chapters we have restricted ourselves in most instances to various finite processes and elementary integration. For an exception see Section 9. As is true in most branches of analysis we make comparatively little progress until we introduce infinite processes. One of the simplest of these is the power series.

We shall consider in some detail a small portion of the known results and methods in the study of linear differential equations with the aid of power series. Before attempting to describe the general situation, which we shall attempt to do only for the second-order linear equations, we first solve a simple numerical example.

Consider

$$\frac{d^2w}{dz^2} + zw = 0.$$

It is within reason, finite processes having failed to yield the solution of this equation, for us to hope that a solution may be found in the form of a power series in z or in the form of such a series multiplied by a power, not necessarily integral or positive, of z. Let us then assume

$$w = \sum_{n=0}^{\infty} a_n z^{n+c}.$$

Since we are assuming the existence of such a solution and since, if it exists, it must have a first term, we may assume that $a_0 \neq 0$.

We substitute this expression for w into our original equation and hope to determine c and the a_n. We have at once

$$\sum_{n=0}^{\infty} (n + c)(n + c - 1)a_n z^{n+c-2} + \sum_{n=0}^{\infty} a_n z^{n+c+1} = 0,$$

which is to be an identity in z. We may now, by shifting the index in the latter summation from n to $(n-3)$, write

$$\sum_{n=0}^{\infty} (n+c)(n+c-1)a_n z^{n+c-2} + \sum_{n=3}^{\infty} a_{n-3} z^{n+c-2} = 0,$$

in which the exponent of z is the same in the two summations. We may combine terms for $n \geqq 3$ and write

$$c(c-1)a_0 z^{c-2} + (c+1)ca_1 z^{c-1} + (c+2)(c+1)a_2 z^c$$
$$+ \sum_{n=3}^{\infty} [(n+c)(n+c-1)a_n + a_{n-3}]z^{n+c-2} = 0.$$

We now refer to the theorem that for a power series to vanish for all values of z in some region (not just a point) it is necessary and sufficient that the coefficient of each power of z in that power series vanish identically. Thus we have the system of equations

$$c(c-1)a_0 = 0,$$
$$(c+1)ca_1 = 0,$$
$$(c+2)(c+1)a_2 = 0,$$

and, for $n \geqq 3$,

$$(n+c)(n+c-1)a_n + a_{n-3} = 0.$$

Since $a_0 \neq 0$, the first equation above becomes

$$c(c-1) = 0.$$

This is called the *indicial equation.* Its counterpart in the general theory will play an important role. From the indicial equation we have $c = 0$ or $c = 1$. From this point on we might divide our work into two sections, in each of them using one of these values for c.

Let us concentrate our attention on $c = 0$. With this value for c the second equation of the above system is automatically satisfied and a_1 is thus seen to be an arbitrary constant. Recalling that the term $a_0 z^c$ is independent of a_1, and that a_0 and a_1 are each arbitrary constants, we see that success in our present attempt should bring with it the general solution of the differential equation. Our system of equations has become

$$a_2 = 0,$$

and, for $n \geqq 3$,

$$a_n = -\frac{a_{n-3}}{n(n-1)}.$$

The last equation is called a *recurrence relation*. From it we may compute each coefficient in our series from a particular preceding coefficient. But we can do much better than that. Since the above recurrence relation holds for all $n \geqq 3$, we may replace n by $(n - 3)$ and find

$$a_{n-3} = -\frac{a_{n-6}}{(n-3)(n-4)}.$$

Then, substituting for a_{n-3} in the original, we find

$$a_n = (-1)^2 \frac{a_{n-6}}{n(n-1)(n-3)(n-4)},$$

and we may use the original formula over and over until we determine a_n in terms of a_0, a_1, or a_2.

Examination of the recurrence relation makes it evident that our a_n divide themselves naturally into three classes according as n is a multiple of 3, one greater than a multiple of 3, or two greater than a multiple of 3. Hence we treat the three cases $n = 3k$, $n = 3k + 1$, $n = 3k + 2$, where k is a positive integer. From $a_2 = 0$ it follows at once that $a_{3k+2} = 0$ for all $k > 0$.

Consider now

$$a_{3k+1} = -\frac{a_{3k-2}}{(3k+1)(3k)}.$$

We may write

$$a_{3k+1} = (-1)^2 \frac{a_{3k-5}}{(3k+1)(3k)(3k-2)(3k-3)}$$

and, by reapplication of the recurrence relation, finally express a_{3k+1} in terms of a_1. The last step in this procedure would involve the use of the relation

$$a_4 = -\frac{a_1}{4 \cdot 3}.$$

Furthermore, we note that in reducing the subscript $(3k + 1)$ to the subscript 1 at the rate of 3 per application we must apply the recurrence relation precisely k times. Thus we have, for $k \geqq 1$,

$$a_{3k+1} = (-1)^k \frac{a_1}{(3k+1)(3k)(3k-2)(3k-3) \cdots 4 \cdot 3}.$$

In this result the denominator contains the product of all factors of the form $(3i)$ and of the form $(3i + 1)$ from $i = 1$ to $i = k$. If we insert $2 \cdot 5 \cdots (3k - 1)$, all factors of the form $(3i - 1)$ from $i = 1$ to $i = k$, in both numerator and denominator, we may write more simply

$$a_{3k+1} = (-1)^k \frac{2 \cdot 5 \cdot 8 \cdots (3k-1)}{(3k+1)!} a_1 .$$

Next we consider the remaining case. From

$$a_{3k} = -\frac{a_{3k-3}}{(3k)(3k-1)},$$

we may, by a procedure parallel to the one used above, find

$$a_{3k} = (-1)^k \frac{a_0}{(3k)(3k-1)(3k-3)(3k-4) \cdots 3 \cdot 2},$$

and, simplifying the denominator as before,

$$a_{3k} = (-1)^k \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k)!} a_0.$$

We are now in a position to write

$$w = a_0 \left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k)!} z^{3k} \right] + a_1 \left[z + \sum_{k=1}^{\infty} (-1)^k \frac{2 \cdot 5 \cdot 8 \cdots (3k-1)}{(3k+1)!} z^{3k+1} \right],$$

the desired general solution of our differential equation.

We are able now to justify our result directly. Let $w_0(z)$ be the coefficient of a_0 and let $w_1(z)$ be the coefficient of a_1 in the above solution. Let us apply the ratio test to the series for $w_0(z)$. We have

$$\lim_{k \to \infty} \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)(3k+1) \, |z|^{3k+3}}{(3k+3)!} \cdot \frac{(3k)!}{1 \cdot 4 \cdot 7 \cdots (3k-2) \, |z|^{3k}} = \lim_{k \to \infty} \frac{|z|^3}{(3k+2)(3k+3)} = 0.$$

Hence the series for w_0 is convergent for all finite z. The same is easily seen to be true of the series for w_1. Now we know that each of the series is unlimitedly differentiable everywhere in the finite plane and therefore that our solution is valid if it formally satisfies the differential equation. That $w = a_0 w_0 + a_1 w_1$ is a formal solution of

$$\frac{d^2w}{dz^2} + zw = 0$$

follows from the manner in which we constructed w_0 and w_1, but we can also check this fact directly. As an example we perform the check for w_0. From

$$w_0 = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k)!} z^{3k}$$

we find

$$\frac{d^2w_0}{dz^2} = \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k-2)!} z^{3k-2},$$

or

$$\frac{d^2w_0}{dz^2} = -z + \sum_{k=2}^{\infty} (-1)^k \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k-2)!} z^{3k-2}$$

$$= -z + \sum_{k=2}^{\infty} (-1)^k \frac{1 \cdot 4 \cdot 7 \cdots (3k-5)}{(3k-3)!} z^{3k-2}.$$

Now let us shift the index in the last summation from k to $(k+1)$. Then we have

$$\frac{d^2w_0}{dz^2} = -z + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k)!} z^{3k+1}.$$

But

$$zw_0 = z + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k)!} z^{3k+1},$$

so that

$$\frac{d^2w_0}{dz^2} + zw_0 = 0.$$

The student should carry through the verification for $w_1(z)$.

If we look back over this section we note that the second solution $c = 1$ of the indicial equation (page 68) has been completely neglected. As another exercise the student should show that $c = 1$ leads exactly to the solution $w_1(z)$ which we have already obtained as a part of the result for $c = 0$.

We close this discussion by calling attention to the fact that w_0 and w_1 are actually linearly independent and that our solution is therefore the complete solution. The linear independence follows at once from the fact that w_0 has a constant term while w_1 has no constant term.

PROBLEMS

1. Solve the equation

$$(1 - \tfrac{1}{2}z^2)\frac{d^2w}{dz^2} + z\frac{dw}{dz} - w = 0.$$

Ans. $w = a_0(1 + \tfrac{1}{2}z^2) + a_1 z.$

2. Solve the equation

$$\frac{d^2w}{dz^2} + z\frac{dw}{dz} + w = 0.$$

Ans.
$$\begin{aligned} w &= a_0\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2 \cdot 4 \cdot 6 \cdots (2k)}\right] \\ &\quad + a_1\left[z + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}\right] \\ &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} z^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!} z^{2k+1} \\ &= a_0 e^{-z^2/2} + a_1 e^{-z^2/2} \int_0^z e^{\alpha^2/2}\, d\alpha. \end{aligned}$$

3. Solve completely

$$\frac{d^2w}{dz^2} + 5z^3 w = 0.$$

Ans.
$$\begin{aligned} w &= a_0\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k z^{5k}}{k!\, 4 \cdot 9 \cdot 14 \cdots (5k-1)}\right] \\ &\quad + a_1\left[z + \sum_{k=1}^{\infty} \frac{(-1)^k z^{5k+1}}{k!\, 6 \cdot 11 \cdot 16 \cdots (5k+1)}\right]. \end{aligned}$$

4. Solve the following equation by means of power series, and check your solution by solving the equation by an elementary method

$$\frac{d^2w}{dz^2} + 4w = 0.$$

26. Multiplication of Series. It will often be necessary for us to use the product of two series. We shall be interested only in what is called the Cauchy product of two series. The product

$$\left(\sum_{n=0}^{\infty} c_n\right)\left(\sum_{n=0}^{\infty} d_n\right)$$

must include all terms of the type $c_i d_j$ for non-negative integral i and j. Therefore we consider the array

$$\begin{array}{l} c_0d_0 + c_0d_1 + c_0d_2 + \cdots + c_0d_n + \cdots \\ + c_1d_0 + c_1d_1 + c_1d_2 + \cdots + c_1d_n + \cdots \\ + c_2d_0 + c_2d_1 + c_2d_2 + \cdots + c_2d_n + \cdots \\ \cdot \\ \cdot \\ \cdot \\ + c_md_0 + c_md_1 + c_md_2 + \cdots + c_md_n + \cdots \\ + \cdots . \end{array}$$

The *Cauchy product* of the original series is the series formed by grouping the above array in finite diagonals. That is, in the Cauchy product, the general term is given by the equation

$$\begin{aligned} p_n &= c_nd_0 + c_{n-1}d_1 + \cdots + c_0d_n \\ &= \sum_{k=0}^{n} c_{n-k}d_k = \sum_{k=0}^{n} c_kd_{n-k}. \end{aligned}$$

Thus we write

$$\left(\sum_{n=0}^{\infty} c_n\right)\left(\sum_{n=0}^{\infty} d_n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_kd_{n-k}. \tag{37}$$

The above definition of the product of two series is useful largely because of the existence of theorems of the type of Cauchy's theorem,*

Theorem 7. If $\sum_{n=0}^{\infty} c_n$ *is absolutely convergent to the sum* C, *and if* $\sum_{n=0}^{\infty} d_n$ *is absolutely convergent to the sum* D, *then* $\sum_{n=0}^{\infty}\sum_{k=0}^{n} c_kd_{n-k}$ *is absolutely convergent to the sum* CD.

Cauchy's theorem is particularly well adapted to the multiplication of the power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(z - z_0)^n,$$

because the Cauchy product of those series is itself a power series.

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* For the proof of this and related theorems (Abel's and Merten's) see L. L. Smail, *Elements of the Theory of Infinite Processes*, 1923, Chapter XI.

That is,

$$\left[\sum_{n=0}^{\infty} a_n(z-z_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(z-z_0)^n\right] = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k(z-z_0)^k b_{n-k}(z-z_0)^{n-k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}(z-z_0)^n.$$

Theorem 7 applies within the smaller of the circles of convergence of the original two power series.

We shall use the summation symbol throughout our work. Once we are thoroughly familiar with that symbol, it seems laborious to carry through the work by writing down a symbol for the general term and then dangling along with it a string of numerical terms and a group of dots. On the other hand, whenever difficulties are met, or until thorough familiarity is attained, we should not hesitate to abandon the compact summation symbol for any increase in clarity which might result from writing, for example,

$$a_0b_0 + (a_0b_1 + a_1b_0)(z-z_0) + (a_0b_2 + a_1b_1 + a_2b_0)(z-z_0)^2 + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 + a_nb_0)(z-z_0)^n + \cdots$$

in place of its equivalent,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}(z-z_0)^n.$$

Practice on a few simple problems in multiplication of series should develop for the student a technique sufficient for our purpose. With this in mind, we include a few examples, comments, and problems.

Let us obtain by multiplication of series an expansion for the function $\dfrac{1}{(1-2z)(1-3z)}$. From

$$\frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n$$

and

$$\frac{1}{1-3z} = \sum_{n=0}^{\infty} 3^n z^n,$$

we find at once that

$$\frac{1}{(1-2z)(1-3z)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} 2^k 3^{n-k} z^n.$$

This is the solution to our problem of multiplication. We do not stop here, of course, when we see some chance of simplifying the expression for the coefficient of z^n in our result. In this case we turn to elementary algebra for the identity

$$\sum_{k=0}^{n} 2^k 3^{n-k} = \frac{3^{n+1} - 2^{n+1}}{3-2} = 3^{n+1} - 2^{n+1}.$$

Hence we have

$$\frac{1}{(1-2z)(1-3z)} = \sum_{n=0}^{\infty} (3^{n+1} - 2^{n+1}) z^n,$$

a result valid in $|z| < \frac{1}{3}$. The result could have been obtained much more easily as follows:

$$\begin{aligned} \frac{1}{(1-2z)(1-3z)} = \frac{3}{1-3z} - \frac{2}{1-2z} &= \sum_{n=0}^{\infty} 3^{n+1} z^n - \sum_{n=0}^{\infty} 2^{n+1} z^n \\ &= \sum_{n=0}^{\infty} (3^{n+1} - 2^{n+1}) z^n. \end{aligned}$$

Our desire was, however, primarily to obtain practice in multiplication of series.

Before proceeding to the next example, we wish to call attention to the fact that the product of a series of even powers by another series of even powers yields a series of even powers. Similar remarks apply when one series has only odd powers and the other only even, or only odd, powers.

Consider next the problem of multiplying the series for $\frac{1}{1-z^2}$,

$$\sum_{n=0}^{\infty} z^{2n},$$

by the series for $\cos z$,

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

We have

$$\begin{aligned}\frac{\cos z}{1-z^2} &= \left[\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{\infty} z^{2n}\right]\\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{1}{(2k)!} z^{2k} \cdot 1 \cdot z^{2n-2k}\\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{1}{(2k)!} z^{2n}.\end{aligned}$$

In this case the coefficient in the general term, namely,

$$\sum_{k=0}^{n} (-1)^k \frac{1}{(2k)!} = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots + (-1)^n \frac{1}{(2n)!},$$

does not yield readily to simplification as did the coefficient in the other example. The above coefficient is easily computed with much accuracy for any n, and that is, after all, our main desire.

Finally we note that in some multiplications the power series involved will need to be rewritten before the Cauchy product will yield the desired result, that is, a power series. If we were to multiply the series for e^z,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and the series for $\cos z$,

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

directly as they stand, the Cauchy product would not give us a power series. Thus

$$\begin{aligned}e^z \cos z &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{z^{2k}}{(2k)!} \cdot \frac{z^{n-k}}{(n-k)!}\\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{1}{(2k)!(n-k)!} z^{n+k}\end{aligned}$$

is not satisfactory because the general term in the result contains different powers of z.

What we need to do is to supply the missing terms in the series for $\cos z$. That is, we need to write

$$\cos z = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = 0$, if n is odd, and $a_n = \dfrac{(-1)^{n/2}}{n!}$, if n is even. Then

$$e^z \cos z = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k z^k \frac{z^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_k}{(n-k)!} z^n,$$

where the a_k are as defined above. This is the desired series.

Refinements of the above result are possible. We may write a single formula for a_n, if we wish. One such formula is

$$a_n = \frac{1 + (-1)^n}{2} \cdot \frac{(-1)^{n/2}}{n!}.$$

Also, we can eliminate the zero terms in the coefficient of z^n in the result. For the real number m, let $[m]$ denote the greatest integer in m; that is, $[m]$ is the largest integer, positive, negative, or zero, which is not greater than m. Then it may be seen that

$$e^z \cos z = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} (-1)^s \frac{1}{(2s)!(n-2s)!} z^n.$$

Fortunately, the type of manipulation used in obtaining this expansion for $e^z \cos z$ is seldom needed by those using only a moderate amount of infinite series. There is only one problem of this kind in the set below.

PROBLEMS

1. Multiply the series for $\dfrac{1}{1+z}$,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

and the series for $\dfrac{1}{1-4z}$,

$$\frac{1}{1-4z} = \sum_{n=0}^{\infty} 4^n z^n.$$

Check your result by using

$$\frac{1}{(1+z)(1-4z)} = \frac{4}{5(1-4z)} + \frac{1}{5(1+z)}.$$

2. Obtain a series for $\dfrac{z \sin z}{1 - z^2}$.

$$\textit{Ans.}\quad \frac{z \sin z}{1 - z^2} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} z^{2n+2}$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)!} z^{2n}.$$

3. Obtain a series for $\dfrac{1}{(1 - z)^2}$ by squaring the series for $\dfrac{1}{1 - z}$. Then check your result by obtaining it by differentiating the series for $\dfrac{1}{1 - z}$.

4. Show that

$$\frac{\log(1+z)}{1-z} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k+1} z^{n+1} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} z^n$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} z^n.$$

5. Obtain the power series for $e^{-z^2} \sin z$.

$$\textit{Ans.}\quad e^{-z^2} \sin z = \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n} \frac{1}{(2k+1)!(n-k)!} z^{2n+1}.$$

6. Multiply the power series for sin z by that for cos z. Recalling that $\sin z \cos z = \frac{1}{2} \sin 2z$, check your result. You will need to prove the identity

$$\sum_{k=0}^{n} \frac{1}{(2k+1)!(2n-2k)!} = \frac{2^{2n}}{(2n+1)!}.$$

This may be done with the aid of the binomial theorem. Note that

$$\sum_{k=0}^{n} \frac{1}{(2k+1)!(2n-2k)!} = \frac{1}{(2n+1)!} \sum_{k=0}^{n} \frac{(2n+1)!}{(2k+1)!(2n-2k)!}$$

$$= \frac{1}{(2n+1)!} \sum_{k=0}^{n} \binom{2n+1}{2k+1},$$

where $\binom{2n+1}{2k+1}$ is the binomial coefficient, which may be defined as the coefficient of x^{2k+1} in the binomial expansion of $(1 + x)^{2n+1}$. Thus prove that

$$\sum_{k=0}^{n} \frac{1}{(2k+1)!(2n-2k)!} = \frac{1}{(2n+1)!} \frac{(1+1)^{2n+1} - (1-1)^{2n+1}}{2}$$

$$= \frac{2^{2n}}{(2n+1)!}.$$

7. Obtain the power series for $\sin^2 z$ from that for sin z by squaring the latter. Then get the desired series from the formula $\sin^2 z = \frac{1}{2}(1 - \cos 2z)$ and, in a manner similar to the procedure of Problem 6, prove that your two results are identical.

8. Obtain the power series for $\dfrac{\sin z}{1 - z}\cdot$

9. Use the series

$$e^{\alpha^2 2} = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{2^n n!}$$

to obtain a series for $\displaystyle\int_0^z e^{\alpha^2/2}\,d\alpha$. Then use multiplication of series to find the power series for

$$e^{-z^2/2} \int_0^z e^{\alpha^2/2}\,d\alpha.$$

Note the connection with Problem 2, Section 25.

27. Ordinary points of Second-Order Linear Equations. The point z_0 in the finite plane is said to be an *ordinary* point of the differential equation

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \tag{38}$$

if, and only if, both $p(z)$ and $q(z)$ are analytic at $z = z_0$. In order to treat the point $z = \infty$, put $z = \dfrac{1}{\zeta}\cdot$ Then $z = \infty$ is called an ordinary point of equation (38) if, and only if, $\zeta = 0$ is an ordinary point of the transformed equation. Since

$$\frac{dw}{dz} = \frac{d\zeta}{dz}\frac{dw}{d\zeta} = -\zeta^2\frac{dw}{d\zeta}$$

and

$$\frac{d^2w}{dz^2} = \zeta^4\frac{d^2w}{d\zeta^2} + 2\zeta^3\frac{dw}{d\zeta},$$

the transformed equation is

$$\frac{d^2w}{d\zeta^2} + \left[\frac{2}{\zeta} - \frac{1}{\zeta^2}p\left(\frac{1}{\zeta}\right)\right]\frac{dw}{d\zeta} + \frac{1}{\zeta^4}q\left(\frac{1}{\zeta}\right)w = 0. \tag{39}$$

That is, we say that $z = \infty$ is an ordinary point of equation (38) if, and only if, both $\left[\dfrac{2}{\zeta} - \dfrac{1}{\zeta^2}p\left(\dfrac{1}{\zeta}\right)\right]$ and $\dfrac{1}{\zeta^4}q\left(\dfrac{1}{\zeta}\right)$ are holomorphic at $\zeta = 0$.

28. Solution Valid Near an Ordinary Point. It is our purpose in this section to solve

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \tag{38}$$

in power series about an ordinary point z_0 of the equation. Since $p(z)$ and $q(z)$ are holomorphic at z_0, we may write

$$p(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n; \quad |z - z_0| < R_1,$$

$$q(z) = \sum_{n=0}^{\infty} d_n(z - z_0)^n; \quad |z - z_0| < R_2,$$

where R_1 and R_2 are respectively the radii of convergence of the series for $p(z)$ and $q(z)$ and where R_1, $R_2 > 0$. Note that all variables and coefficients here, b_n, d_n, etc., are permitted to be complex numbers.

If we assume that

$$w = \sum_{n=0}^{\infty} a_n(z - z_0)^{n+c},$$

we find

$$\sum_{n=0}^{\infty} (n + c)(n + c - 1)a_n(z - z_0)^{n+c-2} + \left[\sum_{n=0}^{\infty} b_n(z - z_0)^n\right]\left[\sum_{n=0}^{\infty} (n + c)a_n(z - z_0)^{n+c-1}\right] + \left[\sum_{n=0}^{\infty} d_n(z - z_0)^n\right]\left[\sum_{n=0}^{\infty} a_n(z - z_0)^{n+c}\right] = 0.$$

Performing the indicated multiplications, we have next

$$\sum_{n=0}^{\infty} (n + c)(n + c - 1)a_n(z - z_0)^{n+c-2} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} (k + c)a_k b_{n-k}(z - z_0)^{n+c-1} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k d_{n-k}(z - z_0)^{n+c} = 0.$$

Let us now shift indices and write

$$\sum_{n=0}^{\infty} (n + c)(n + c - 1)a_n(z - z_0)^{n+c-2} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (k + c)a_kb_{n-k-1}(z - z_0)^{n+c-2} + \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} a_kd_{n-k-2}(z - z_0)^{n+c-2} = 0.$$

This form of the equation permits us to collect terms so that we have

$$c(c - 1)a_0 (z - z_0)^{c-2} + (1 + c)ca_1(z - z_0)^{c-1} + ca_0b_0(z - z_0)^{c-1} + \sum_{n=2}^{\infty} \left[(n + c)(n + c - 1)a_n + \sum_{k=0}^{n-1} (k + c)a_kb_{n-k-1} + \sum_{k=0}^{n-2} a_kd_{n-k-2} \right] (z - z_0)^{n+c-2} = 0.$$

We now set the coefficients of the various powers of $(z - z_0)$ equal to zero. Hence

$$c(c - 1)a_0 = 0,$$

$$c[(1 + c)a_1 + a_0b_0] = 0,$$

and, for $n \geqq 2$,

$$(n + c)(n + c - 1)a_n + \sum_{k=0}^{n-1} (k + c)a_kb_{n-k-1} + \sum_{k=0}^{n-2} a_kd_{n-k-2} = 0.$$

Again we may insist that $a_0 \neq 0$. Thus we have $c = 0$ or $c = 1$. Let us choose $c = 0$. Then the second of the above relations is satisfied automatically so that a_1 is an arbitrary constant. For $n \geqq 2$, a_n is determined by

$$n(n - 1)a_n = -\sum_{k=1}^{n-1} ka_kb_{n-k-1} - \sum_{k=0}^{n-2} a_kd_{n-k-2}.$$

We know (see page 62) that there exist positive constants r_1, r_2, M_1, and M_2, such that

$$|b_n| < \frac{M_1}{r_1^n}, \quad |d_n| < \frac{M_2}{r_2^n}.$$

Next we choose $r = \text{Min}(r_1, r_2)$. That is $r \leqq r_1$, $r \leqq r_2$, with equality in at least one case. We introduce another constant $M > \text{Max}(M_1, M_2, r, |a_1|r, 1)$; that is $M > M_1$, $M > M_2$, $M > r$, $M > |a_1|r$, $M > 1$. We shall also make use of the fact that, if A and B are any complex numbers, $|A + B| \leqq |A| + |B|$.

In terms of our new constants M and r we have

$$|b_n| < \frac{M}{r^n}, \quad |d_n| < \frac{M}{r^n}, \quad |a_1| < \frac{M}{r}.$$

We shall proceed by induction. Let us assume that, for each integral k such that $1 \leqq k < n$, it is true that $|a_k| < \frac{M^k}{r^k}$. Since we already have $|a_1| < \frac{M}{r}$, we need show only that $|a_n| < \frac{M^n}{r^n}$ in order to complete the induction. Now

$$|a_n| \leqq \frac{\sum_{k=1}^{n-1} k|a_k| \cdot |b_{n-k-1}| + \sum_{k=0}^{n-2} |a_k| \cdot |d_{n-k-2}|}{n(n-1)}.$$

Because of our inequalities on $|b_n|$ and $|d_n|$ and the assumed inequality on $|a_k|$ for $1 \leqq k < n$, we have at once

$$|a_n| < \frac{\sum_{k=1}^{n-1} k \frac{M^k}{r^k} \frac{M}{r^{n-k-1}} + \sum_{k=0}^{n-2} \frac{M^k}{r^k} \frac{M}{r^{n-k-2}}}{n(n-1)}.$$

Since, by definition, $M > 1$ and $\frac{M}{r} > 1$, we may introduce additional factors M and $\frac{M}{r}$ wherever needed in the numerator in order to simplify this inequality to the form

$$|a_n| < \frac{\sum_{k=1}^{n-1} k \frac{M^n}{r^n} + \sum_{k=0}^{n-2} \frac{M^n}{r^n}}{n(n-1)},$$

or

$$|a_n| < \frac{\frac{1}{2}n(n-1) + n - 1}{n(n-1)} \cdot \frac{M^n}{r^n}.$$

Thus we find

$$|a_n| < \frac{n+2}{2n} \cdot \frac{M^n}{r^n}.$$

But, for $n \geqq 2$, this leads to $|a_n| < \frac{M^n}{r^n}$, which is what we wished to prove. We now know that there exist positive numbers M and r such that, for each $n \geqq 1$, $|a_n| < \left(\frac{M}{r}\right)^n$. This means that the series

$$w = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

has a radius of convergence at least as large as $\frac{r}{M}$. Furthermore, we have seen that both a_0 and a_1 are arbitrary.

We have proved

Theorem 8. If z_0 *is an ordinary point of the equation*

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0,$$

then there exists a series

$$w = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

(a) *with a non-zero radius of convergence, and* (b) *containing two arbitrary constants* a_0 *and* a_1, *and* (c) *satisfying the differential equation within the circle of convergence of the series.*

Indeed, by our method of proof we have given a means of constructing as many terms as may be desired of the infinite series. The modification needed for an ordinary point at infinity is again merely the use of the transformation $z = \frac{1}{\zeta}$ and the application of Theorem 8 to the new equation.

It is worth noting that directly analogous definitions, proofs, and theorems exist for linear differential equations of any finite order.

29. Singularities of Second-Order Linear Equations. If the point z_0, finite or infinite, is not an ordinary point of a differential equation, we shall call it a *singular point*, or a *singularity*, of the differential equation.

It is necessary to distinguish between singular points of a linear differential equation and singular points of a solution of that equation. It follows from Theorem 8 with a little additional argument that the singular points of any solution of a linear differential equation are contained in the set of singular points of the equation. But it is not true that a singular point of a differential equation is necessarily a singular point of a particular, or of the general, solution of that equation. As an example consider the equation

$$z^2 \frac{d^2w}{dz^2} - 2z \frac{dw}{dz} + 2w = 0,$$

which has $z = 0$ as a singular point. The general solution of this equation is $w = b_0 z + b_1 z^2$, where b_0 and b_1 are arbitrary constants. Evidently every solution of this equation is analytic at $z = 0$. In fact, the solutions have singularities at the one point $z = \infty$, which is the only other singular point of the equation.

The preceding paragraph suggests a better result than was given for the radius of convergence of the series in Theorem 8, page 83. We see now that the circle of convergence of the series solution of Theorem 8 about an ordinary point of a differential equation extends at least as far as the nearest singularity of the differential equation. The equation used above furnishes us with an example in which the region of convergence extends even beyond the nearest singularity of the differential equation. The series

$$w(z) = a_0 + a_1(z - 1) + (a_1 - a_0)(z - 1)^2,$$

where a_0 and a_1 are arbitrary constants, is a solution of

$$z^2 \frac{d^2w}{dz^2} - 2z \frac{dw}{dz} + 2w = 0$$

valid about the point $z = 1$, an ordinary point of the differential equation. The series is convergent for all finite z, although $z = 0$ is a singular point of the differential equation.

We know that linear differential equations are found in great abundance in the problems of applied mathematics. Indeed, it seems that very many of these problems lead to equations of the second order. Another interesting observation is that in applied problems very often we are interested precisely in solutions of the linear differential equation valid in the neighborhood of a singular point of the differential equation.

Before we proceed to study solutions in the neighborhood of singular points it becomes wise for us to classify singular points. We shall devote most of our attention to a simple type called a regular singular point.

DEFINITION 3. The point z_0, in the finite plane, is called a *regular singular point* of the equation

$$\frac{d^2w}{dz^2}+\frac{P(z)}{z-z_0}\frac{dw}{dz}+\frac{Q(z)}{(z-z_0)^2}w=0 \tag{40}$$

if, and only if, conditions (a) and (b) are both satisfied: (a) z_0 is not an ordinary point of the equation; (b) $P(z)$ and $Q(z)$ are holomorphic at z_0.

As usual, we say that $z = \infty$ is a regular singular point of equation (40) if, and only if, when the equation is transformed by $z = \frac{1}{\zeta}$ it is found that $\zeta = 0$ is a regular singular point of the transformed equation.

If z_0 is a singular point of a linear differential equation, but is not a regular singular point of that equation, we then call z_0 an *irregular singular point* of the equation. It seems evident that for detailed study of irregular singular points a finer classification would be advisable. We shall restrict ourselves to the study of solutions valid near a regular singular point.

PROBLEMS

Locate and classify all singular points of each of the following equations:

1. $z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-n^2)w=0$; n constant (Bessel's equation of index n).

Ans. Regular singularity at $z = 0$, irregular one at $z = \infty$.

2. $(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+n(n+1)w=0$; n constant (Legendre's equation).

Ans. Regular singularities at $z = 1, -1, \infty$.

3. $z\frac{d^2w}{dz^2}+(\nu+1)\frac{dw}{dz}+w=0$; ν constant (Bessel-Clifford equation).

4. $z(1-z)\frac{d^2w}{dz^2}+[\gamma-(\alpha+\beta+1)z]\frac{dw}{dz}-\alpha\beta w=0$; α, β, γ constants (the hypergeometric equation of Gauss).

5. $\dfrac{d^2w}{dz^2} + (a + 16q \cos 2z)w = 0$; a, q constants (Mathieu's equation).

6. $4z(1 - z)\dfrac{d^2w}{dz^2} + 2(1 - 2z)\dfrac{dw}{dz} + (a - 16q + 32qz)w = 0$; a, q constants (Lindemann's form of Mathieu's equation).

7. $\dfrac{d^2w}{dz^2} + \left[-\dfrac{1}{4} + \dfrac{k}{z} + \dfrac{\frac{1}{4} - m^2}{z^2}\right]w = 0$; k, m constants (Whittaker's confluent hypergeometric equation).

8. $z\dfrac{d^2w}{dz^2} + (\gamma - z)\dfrac{dw}{dz} - \alpha w = 0$; α, γ constants (another confluent hypergeometric equation).

9. $\dfrac{d^2w}{dz^2} + k^2w = 0$; k constant.

10. $z^2\dfrac{d^2w}{dz^2} + w = 0$.

30. Solution Valid Near a Regular Singular Point. We shall discuss now the solution by means of power series about z_0 of the equation

$$\frac{d^2w}{dz^2} + \frac{P(z)}{z - z_0}\frac{dw}{dz} + \frac{Q(z)}{(z - z_0)^2}w = 0, \tag{40}$$

under the assumption that z_0 is a regular singular point of the equation. Under this assumption $P(z)$ and $Q(z)$ are holomorphic at z_0, and we have

$$P(z) = \sum_{n=0}^{\infty} p_n(z - z_0)^n; \quad |z - z_0| < R_1,$$

$$Q(z) = \sum_{n=0}^{\infty} q_n(z - z_0)^n; \quad |z - z_0| < R_2,$$

where $R_1 > 0$ and $R_2 > 0$ are the radii of convergence of the respective power series. The fact that z_0 is not an ordinary point of equation (40) is equivalent to the statement that p_0, q_0, and q_1 are not all zero.

First we assume a solution in the form

$$w = \sum_{n=0}^{\infty} a_n(z - z_0)^{n+c},$$

where $a_0 \neq 0$ is arbitrary and c and the remaining a_n's are to be determined. Direct substitution into equation (40) yields

$$\sum_{n=0}^{\infty} (n+c)(n+c-1)a_n(z-z_0)^{n+c-2} + \left[\sum_{n=0}^{\infty} p_n(z-z_0)^n\right]\left[\sum_{n=0}^{\infty} (n+c)a_n(z-z_0)^{n+c-2}\right] + \left[\sum_{n=0}^{\infty} q_n(z-z_0)^n\right]\left[\sum_{n=0}^{\infty} a_n(z-z_0)^{n+c-2}\right] = 0.$$

Next we have

$$\sum_{n=0}^{\infty} (n+c)(n+c-1)a_n(z-z_0)^{n+c-2} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} (k+c)a_k p_{n-k}(z-z_0)^{n+c-2} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k q_{n-k}(z-z_0)^{n+c-2} = 0,$$

or

$$\sum_{n=0}^{\infty}\left\{(n+c)(n+c-1)a_n + \sum_{k=0}^{n} a_k[(k+c)p_{n-k} + q_{n-k}]\right\}(z-z_0)^{n+c-2} = 0.$$

We are thus led to the following equations: for $n = 0$,

$$[c(c-1) + cp_0 + q_0]a_0 = 0,$$

and, for $n \geqq 1$,

$$[(n+c)(n+c-1) + (n+c)p_0 + q_0]a_n = -\sum_{k=0}^{n-1} a_k[(k+c)p_{n-k} + q_{n-k}],$$

in which a_n is expressed in terms of a_k; $0 \leqq k < n$. Since $a_0 \neq 0$, we have

$$c^2 + (p_0 - 1)c + q_0 = 0,$$

the important *indicial equation* for the determination of c. The two roots of this equation are called the *exponents* of the differential equation at the regular singularity z_0. For convenience we shall define

$$F(c) = c^2 + (p_0 - 1)c + q_0.$$

Then, of course, the indicial equation becomes $F(c) = 0$, but also the general recurrence relation for a_n; $n \geqq 1$, becomes

$$F(n + c)a_n = - \sum_{k=0}^{n-1} a_k[\,(k + c)p_{n-k} + q_{n-k}\,].$$

Let us call the roots of the indicial equation c_1 and c_2 and choose the notation so that $R(c_1) \geqq R(c_2)$, where the notation $R(z)$ means the "real part" x of the complex number $z = x + iy$. Similarly we use the notation $I(z) = y$ to mean the coefficient of the "imaginary part" of the complex number z. Then let us define $s = c_1 - c_2$. Of course, $R(s) \geqq 0$. Finally we note that we may write $F(c)$ in terms of the exponents or roots of the indicial equation. In fact, $F(c) = (c - c_1)(c - c_2)$.

We now choose to use the exponent c_1; that is, everywhere we shall place $c = c_1$. Consider $F(n + c_1)$. We find that

$$\begin{aligned} F(n + c_1) &= (n + c_1 - c_1)(n + c_1 - c_2) \\ &= n(n + c_1 - c_2) = n(n + s). \end{aligned}$$

Our recurrence relation has become

$$n(n + s)a_n = - \sum_{k=0}^{n-1} a_k[\,(k + c_1)p_{n-k} + q_{n-k}\,],$$

and we wish to obtain an inequality for $|a_n|$ similar to the one derived in Section 28.

We know that there exist positive constants r_1, r_2, M_1, and M_2, such that $|p_n| < \dfrac{M_1}{r_1^n}$ and $|q_n| < \dfrac{M_2}{r_2^n}$. If we take $r = \text{Min}\,(r_1, r_2)$, we have

$$|p_n| < \frac{M_1}{r^n}, \quad |q_n| < \frac{M_2}{r^n},$$

and thus

$$|c_1p_n + q_n| \leqq |c_1| \cdot |p_n| + |q_n| < \frac{|c_1|\,M_1 + M_2}{r^n}.$$

Now let us choose $M = \text{Max}\,(M_1, M_2, |c_1|\,M_1 + M_2, 1)$. Then

$$|p_n| < \frac{M}{r^n}, \quad |q_n| < \frac{M}{r^n}, \quad |c_1p_n + q_n| < \frac{M}{r^n}.$$

The recurrence relation for $n = 1$ is

$$1 \cdot (1 + s)a_1 = -a_0(c_1p_1 + q_1).$$

We choose $a_0 = 1$ and at the same time note that

$$|1 + s| = \sqrt{[1 + R(s)]^2 + [I(s)]^2} \geqq 1 + R(s) \geqq 1.$$

Then we have

$$|a_1| = \frac{|c_1p_1 + q_1|}{|1 + s|} < \frac{M}{r}.$$

We next assume that $|a_k| < \dfrac{M^k}{r^k}$ holds for all integral k such that $1 \leqq k < n$, as we know it does for $k = 1$. We complete the induction by proving that with this assumption $|a_n| < \dfrac{M^n}{r^n}$ follows. We have

$$|a_n| = \frac{\left|\sum\limits_{k=0}^{n-1} a_k[(k + c_1)p_{n-k} + q_{n-k}]\right|}{n\,|n + s|}$$

$$= \frac{\left|\sum\limits_{k=0}^{n-1} ka_kp_{n-k} + \sum\limits_{k=0}^{n-1} a_k(c_1p_{n-k} + q_{n-k})\right|}{n\,|n + s|},$$

$$|a_n| \leqq \frac{\sum\limits_{k=0}^{n-1} k|a_k| \cdot |p_{n-k}| + \sum\limits_{k=0}^{n-1} |a_k| \cdot |c_1p_{n-k} + q_{n-k}|}{n\,|n + s|}.$$

But $|n + s| \geqq n$, since $R(s) \geqq 0$, and our other inequalities may be used to reduce the one on $|a_n|$ to

$$|a_n| < \frac{\sum\limits_{k=0}^{n-1} k\,\dfrac{M^k}{r^k} \cdot \dfrac{M}{r^{n-k}} + \sum\limits_{k=0}^{n-1} \dfrac{M^k}{r^k} \cdot \dfrac{M}{r^{n-k}}}{n^2}.$$

Again we use the fact that $M \geqq 1$ to write

$$|a_n| < \frac{\sum\limits_{k=0}^{n-1} k\,\dfrac{M^n}{r^n} + \sum\limits_{k=0}^{n-1} \dfrac{M^n}{r^n}}{n^2},$$

or

$$|a_n| < \frac{\frac{1}{2}n(n-1)+n}{n^2}\cdot\frac{M^n}{r^n}.$$

Thus

$$|a_n| < \frac{n+1}{2n}\cdot\frac{M^n}{r^n},$$

so that, for $n \geqq 1$, $|a_n| < \frac{M^n}{r^n}$ follows.

We have proved that, *if* z_0 *is a regular singular point of equation (40) and if the indicial equation has its roots* c_1 *and* c_2 *so designated that* $R(c_1) \geqq R(c_2)$, *then there exists a series*

$$w_1(z) = (z - z_0)^{c_1} + \sum_{n=1}^{\infty} a_n(z - z_0)^{n+c_1},$$

satisfying equation (40) within the region of convergence of the series and having a non-zero radius of convergence. It is also worth noting that we have obtained recurrence formulas by which the successive a_n's may be computed.

We are now confronted with the problem of obtaining a second solution linearly independent of $w_1(z)$. It is natural to attempt to use $c = c_2$. *This succeeds if* $s = c_1 - c_2$ *is neither zero nor a positive integer and may succeed even if* s *is a positive integer.*

Let us examine the situation with $c = c_2$. We have

$$\begin{aligned} F(n + c_2) &= (n + c_2 - c_1)(n + c_2 - c_2) \\ &= n(n + c_2 - c_1) = n(n - s). \end{aligned}$$

Thus our recurrence relation (page 88) becomes

$$n(n - s)a_n = -\sum_{k=0}^{n-1} a_k[\,(k + c_2)p_{n-k} + q_{n-k}\,].$$

If s is neither zero nor a positive integer, everything goes through as before except that another constant, in addition to M and r, must be introduced in treating the inequality for $|a_n|$. The details in the proof are omitted.

If $s = 0$, the steps involving c_2 will not differ from the corresponding steps involving c_1, hence no new solution will be obtained.

If s is a positive integer, the recurrence relation for a_s either will be impossible or will be automatically satisfied, leaving a_0 and a_s arbitrary. If that relation is impossible, no new solution is obtained. When the relation is automatically satisfied, a new solution, indeed the general solution, comes out of the work involving c_2.

We shall depend upon experience gained in numerical examples for our treatment of equations leading to these exceptional values of s. There will be no general treatment in this book.

31. Numerical Example: Case I. ***s* neither zero nor an integer.** In this section we treat an unexceptional case. Consider

$$2z\frac{d^2w}{dz^2} + \frac{dw}{dz} - w = 0. \tag{41}$$

First, using the form

$$\frac{d^2w}{dz^2} + \frac{\frac{1}{2}}{z}\frac{dw}{dz} + \frac{-z/2}{z^2}w = 0,$$

we see that $z = 0$ is a regular singular point of equation (41) and that it is the only singular point of that equation in the finite part of the plane. Using $z = \frac{1}{\zeta}$, we arrive at

$$\frac{d^2w}{d\zeta^2} + \frac{\frac{3}{2}}{\zeta}\frac{dw}{d\zeta} + \frac{-1/(2\zeta)}{\zeta^2}w = 0,$$

from which we see that $z = \infty$ is an irregular singular point of equation (41). Now that we have characterized all the singular points of the equation we proceed to study the solutions in the neighborhood of the regular singular point $z = 0$.

For $z = 0$ the functions in the standard form (40) on page 86 are $P(z) = \frac{1}{2}$, $Q(z) = -\frac{1}{2}z$. Hence $p_0 = \frac{1}{2}$ and $q_0 = 0$. Then our indicial equation is

$$c^2 + (\tfrac{1}{2} - 1)c + 0 = 0,$$

or

$$c^2 - \tfrac{1}{2}c = 0,$$

so that $c_1 = \frac{1}{2}$, $c_2 = 0$, and $s = \frac{1}{2}$, which is not integral.

We now use $c = c_1 = \frac{1}{2}$ and assume that

$$w = \sum_{n=0}^{\infty} a_n z^{n+(1/2)}.$$

We shall substitute w into equation (41) in order to determine the a_n. Here two facts are worth noting. First, we can obtain the recurrence relation for our a_n by employing the formulas developed in Section 30. This is not wise. It requires that we make reference to complicated general formulas for the sole purpose of avoiding a relatively simple numerical procedure. Further, it may lead us into an unnecessary multiplication of infinite series. A second fact worth noting is basically identical with the remark in the previous sentence. It is that *whenever the coefficients in a linear equation are rational functions, the multiplication of infinite series may be avoided by clearing of fractions before proceeding to the solution.* It happens not to matter in this particular problem. Hence we see that, once we leave the pure theoretical discussion, the standard form (40) may not be the best form of the equation for us to use.

Let us return to our problem. We have by direct substitution

$$\sum_{n=0}^{\infty} 2(n+\tfrac{1}{2})(n-\tfrac{1}{2})a_n z^{n-(1/2)} + \sum_{n=0}^{\infty} (n+\tfrac{1}{2})a_n z^{n-(1/2)} - \sum_{n=0}^{\infty} a_n z^{n+(1/2)} = 0.$$

Combining the first two series and shifting index in the last, we find

$$\sum_{n=0}^{\infty} (n+\tfrac{1}{2})(2n)a_n z^{n-(1/2)} - \sum_{n=1}^{\infty} a_{n-1} z^{n-(1/2)} = 0.$$

Hence

$$\sum_{n=1}^{\infty} [\,n(2n+1)a_n - a_{n-1}\,] z^{n-(1/2)} = 0.$$

Thus, for $n \geqq 1$,

$$a_n = \frac{a_{n-1}}{n(2n+1)} = \frac{2a_{n-1}}{(2n+1)(2n)}.$$

We proceed to solve the recurrence relation for a_n. We find that

$$a_n = \frac{2a_{n-1}}{(2n+1)(2n)} = \frac{2^2 a_{n-2}}{(2n+1)(2n)(2n-1)(2n-2)}$$
$$= \cdots = \frac{2^n a_0}{(2n+1)(2n)(2n-1)\cdots 4\cdot 3\cdot 2},$$

where the last form of the recurrence relation used was

$$a_1 = \frac{2a_0}{3 \cdot 2}.$$

We now have

$$a_n = \frac{2^n a_0}{(2n+1)(2n)(2n-1)\cdots 4 \cdot 3 \cdot 2} = \frac{2^n}{(2n+1)!} a_0.$$

That is,

$$w = a_0 \sum_{n=0}^{\infty} \frac{2^n z^{n+(1/2)}}{(2n+1)!} = a_0 w_1(z).$$

In order to get another solution, we use $c = c_2 = 0$ and put

$$w = \sum_{n=0}^{\infty} b_n z^n.$$

Then we have

$$\sum_{n=2}^{\infty} 2n(n-1)b_n z^{n-1} + \sum_{n=1}^{\infty} n b_n z^{n-1} - \sum_{n=0}^{\infty} b_n z^n = 0,$$

or

$$b_1 - b_0 + \sum_{n=2}^{\infty} n(2n-1)b_n z^{n-1} - \sum_{n=2}^{\infty} b_{n-1} z^{n-1} = 0.$$

Hence our relations among the b_n become $b_1 = b_0$, and, for $n \geqq 2$,

$$b_n = \frac{b_{n-1}}{n(2n-1)} = \frac{2b_{n-1}}{(2n)(2n-1)}.$$

Proceeding as we did with the a_n, we find

$$b_n = \frac{2^2 b_{n-2}}{(2n)(2n-1)(2n-2)(2n-3)}$$
$$= \cdots = \frac{2^{n-1} b_1}{(2n)(2n-1)(2n-2)\cdots 5 \cdot 4 \cdot 3} = \frac{2^n}{(2n)!} b_0.$$

Finally we have

$$w = b_0 \sum_{n=0}^{\infty} \frac{2^n z^n}{(2n)!} = b_0 w_2(z).$$

Then the general solution of equation (41) is $w = a_0 w_1 + b_0 w_2$, where a_0 and b_0 are arbitrary constants. We have completed the solution. Incidentally we note that the series for w_1 and w_2 converge for all finite z, so that our solution is valid in every region in the finite z plane.

As a remark entirely outside the scope of our present work, we note that

$$w_2 = \sum_{n=0}^{\infty} \frac{2^n z^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(2z)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(\sqrt{2z})^{2n}}{(2n)!},$$

so that $w_2 = \cosh \sqrt{2z}$. Further,

$$w_1 = \sum_{n=0}^{\infty} \frac{2^n z^{n+(1/2)}}{(2n+1)!} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(2z)^{n+(1/2)}}{(2n+1)!} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(\sqrt{2z})^{2n+1}}{(2n+1)!},$$

so that $w_1 = 2^{-1/2} \sinh \sqrt{2z}$, and our general solution could have been written

$$w = A \cosh \sqrt{2z} + B \sinh \sqrt{2z},$$

where A and $B = \dfrac{a_0}{\sqrt{2}}$ are arbitrary constants.

The fact that we were able to recognize the functions represented in our solution may be very useful in actual applications whenever sufficiently good tables of those functions exist. It may save considerable computation. From a theoretical standpoint, however, this may be considered as an interesting but relatively unimportant detail. It means merely that these particular series or functions have occurred so frequently and the properties of these functions have been studied in such detail that we would have been able to recognize the solution, had the equation been put in a slightly different form (with $t = \sqrt{2z}$ as a new independent variable).

It is not our intention to attempt to detract from the credit due to investigations showing that certain equations lead to functions already known, and hence link themselves with other investigations. We are emphasizing the fact that the series solutions above would have been just as useful had they never been seen before.

The student will have noticed that in this section we have defined two particular solutions of our equation as $w_1(z)$ and $w_2(z)$. That notation is an example of a general policy which we shall follow hereafter, a policy which we now set forth explicitly. We shall denote by $w_1(z)$ that particular solution which corresponds to the exponent c_1 and which has for its first term precisely $(z - z_0)^{c_1}$. Another particular solution $w_2(z)$ is to be similarly associated with the exponent c_2. If $c_1 = c_2$, we shall also denote two linearly independent particular solutions by $w_1(z)$, corresponding to the exponent c_1, and by $w_2(z)$. The definition of the latter will be made unique when we first encounter such a solution.

PROBLEMS

1. Solve the equation

$$4z\frac{d^2w}{dz^2} + 2(1 - z)\frac{dw}{dz} - w = 0.$$

Ans.

$$w = a_0 \sum_{n=0}^{\infty} \frac{z^{n+(1/2)}}{1 \cdot 3 \cdot 5 \cdots (2n + 1)} + b_0 \sum_{n=0}^{\infty} \frac{z^n}{2^n n!}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{2^n n! z^{n+(1/2)}}{(2n + 1)!} + b_0 e^{z/2}$$

$$= Ae^{z/2} \int_0^{\sqrt{z/2}} e^{-\alpha^2}\, d\alpha + Be^{z/2}.$$

2. Solve the equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^3 - 2)w = 0.$$

Ans.

$$w = a_0 z^{\sqrt{2}}\left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{z^{3k}}{3^k k!(3+2\sqrt{2})(6+2\sqrt{2}) \cdots (3k+2\sqrt{2})}\right]$$

$$+ b_0 z^{-\sqrt{2}}\left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{z^{3k}}{3^k k!(3-2\sqrt{2})(6-2\sqrt{2}) \cdots (3k-2\sqrt{2})}\right].$$

3. Solve the equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \tfrac{1}{9})w = 0.$$

4. Solve the equation

$$3(z^2 + z)\frac{d^2w}{dz^2} + (z + 2)\frac{dw}{dz} - w = 0$$

in the neighborhood of $z = 0$.

Ans. $w = a_0 w_1(z) + b_0 w_2(z)$.

$$w_1(z) = z^{1/3} + 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{3^n(3n - 2)(3n + 1)n!} z^{n+(1/3)}.$$

$$w_2(z) = 1 + \tfrac{1}{2}z.$$

5. Solve the equation of Problem 4 in the neighborhood of $z = -1$. It is perhaps wise first to introduce a new independent variable by means of the relation $z = t - 1$, and then to solve the resultant differential equation in the neighborhood of $t = 0$. Thus we find that

$$3t(t-1)\frac{d^2w}{dt^2} + (t+1)\frac{dw}{dt} - w = 0,$$

from which

$$w_1(t) = t^{4/3} + 4\sum_{n=1}^{\infty} \frac{5 \cdot 8 \cdot 11 \cdots (3n+2)}{3^n(3n+1)(3n+4)n!} t^{n+(4/3)}$$

and

$$w_2(t) = 1 + t.$$

Hence, finally $w = Aw_1(z) + Bw_2(z)$, where

$$w_1(z) = (z+1)^{4/3} + 4\sum_{n=1}^{\infty} \frac{5 \cdot 8 \cdot 11 \cdots (3n+2)}{3^n(3n+1)(3n+4)n!} (z+1)^{n+(4/3)},$$

$$w_2(z) = 1 + (z+1).$$

6. Solve the equation

$$2z^2\frac{d^2w}{dz^2} + (z^3 - 3z)\frac{dw}{dz} + 2w = 0.$$

$$\textit{Ans.}\quad w_1(z) = z^2 + \sum_{k=1}^{\infty} (-1)^k \frac{z^{2k+2}}{7 \cdot 11 \cdot 15 \cdots (4k+3)}.$$

$$w_2(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+(1/2)}}{4^k k!} = z^{1/2}e^{-z^2/4}.$$

7. Solve the equation

$$8z^2\frac{d^2w}{dz^2} + 2z\frac{dw}{dz} + (1 - z)w = 0.$$

32. Numerical Example: Case II. $s = 0$. We study now the first exceptional case. For this we choose an equation whose solutions are sometimes called modified Bessel* functions of index zero or Bessel functions of pure imaginary argument and index zero. Consider

$$z\frac{d^2w}{dz^2} + \frac{dw}{dz} - zw = 0, \tag{42}$$

an equation which has a regular singular point at $z = 0$ and an irregular one at $z = \infty$.

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* Friedrich Wilhelm Bessel (1784–1846) was a German applied mathematician. Leonhard Euler (1707–1783), born in Switzerland, one of the greatest mathematicians in the history of the subject, used Bessel functions as early as 1764.

Let us obtain solutions valid in the neighborhood of $z = 0$. In our standard notation, $P(z) = 1$ and $Q(z) = -z^2$. Hence $p_0 = 1$ and $q_0 = 0$, so that our indicial equation becomes $c^2 = 0$ and we have $c_1 = c_2 = 0$, $s = 0$. We put

$$w = \sum_{n=0}^{\infty} a_n z^n$$

and arrive at the equation

$$\sum_{n=0}^{\infty} n(n-1)a_n z^{n-1} + \sum_{n=0}^{\infty} n a_n z^{n-1} - \sum_{n=0}^{\infty} a_n z^{n+1} = 0.$$

Hence

$$a_1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} - \sum_{n=2}^{\infty} a_{n-2} z^{n-1} = 0.$$

Next we write

$$a_1 + \sum_{n=2} (n^2 a_n - a_{n-2}) z^{n-1} = 0,$$

from which we find $a_1 = 0$ and, for $n \geqq 2$,

$$a_n = \frac{a_{n-2}}{n^2}.$$

From $a_1 = 0$ it follows that $a_{2k+1} = 0$ for all integral $k > 0$. We are left with a_{2k} for $k \geqq 1$. Putting $n = 2k$ in the above recurrence relation and successively reducing the subscript on the right, we have

$$a_{2k} = \frac{a_{2k-2}}{(2k)^2} = \frac{a_{2k-4}}{(2k)^2(2k-2)^2} = \cdots = \frac{a_0}{(2k)^2(2k-2)^2 \cdots (4)^2(2)^2}.$$

But in the last form of a_{2k} we have the product of the squares of all even numbers from 2 to $(2k)$. If we factor 2^2 out of each of these factors, we obtain the factor 2^{2k} multiplied by the square of the product of all integers from 1 to k. Hence, for $k \geqq 1$,

$$a_{2k} = \frac{a_0}{2^{2k}(k!)^2}.$$

We have found that

$$w = a_0\left[1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{2^{2k}(k!)^2}\right] = a_0 \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(k!)^2} = a_0 w_1(z).$$

This $w_1(z)$ is the function usually denoted by $I_0(z)$.

In an attempt to get a second solution of our differential equation we reason as follows. The double root $c = 0$ of the indicial equation contains the essence of the exceptional character of the differential equation as compared to Case I. Once we actually put $c = 0$ we lose all chance for getting two linearly independent solutions. Therefore we shall assume that

$$w_c = z^c + \sum_{n=1}^{\infty} b_n z^{n+c}, \tag{43}$$

and retain the c as a parameter. We shall substitute this expression in the left member of the equation,

$$z\frac{d^2w}{dz^2} + \frac{dw}{dz} - zw = 0, \tag{42}$$

and determine the manner in which putting $c = 0$ makes that left member vanish.

First we have

$$\begin{aligned} z\frac{d^2w_c}{dz^2} + \frac{dw_c}{dz} - zw_c &= c(c-1)z^{c-1} \\ &\quad + \sum_{n=1}^{\infty} (n+c)(n+c-1)b_n z^{n+c-1} + cz^{c-1} \\ &\quad + \sum_{n=1}^{\infty} (n+c)b_n z^{n+c-1} - z^{c+1} - \sum_{n=1}^{\infty} b_n z^{n+c+1} \\ &= c^2z^{c-1} + \sum_{n=1}^{\infty} (n+c)^2 b_n z^{n+c-1} - z^{c+1} - \sum_{n=3}^{\infty} b_{n-2} z^{n+c-1}. \end{aligned}$$

We may combine the summations and write

$$\begin{aligned} z\frac{d^2w_c}{dz^2} + \frac{dw_c}{dz} - zw_c &= c^2z^{c-1} + (c+1)^2 b_1 z^c + [\,(c+2)^2 b_2 - 1\,]z^{c+1} \\ &\quad + \sum_{n=3}^{\infty} [\,(n+c)^2 b_n - b_{n-2}\,]z^{n+c-1}. \end{aligned}$$

As explained before, we do not wish to choose c at this point. We are able, however, to make all but the first term of the right-hand member of the above equation vanish without selecting c. To do this we must take $b_1 = 0$, $b_2 = \dfrac{1}{(c+2)^2}$, and, for $n \geqq 3$,

$$b_n = \frac{b_{n-2}}{(n+c)^2}.$$

Following methods used earlier in this section we find that $b_{2k+1} = 0$ for all integral $k \geqq 0$ and that

$$b_{2k} = \frac{b_{2k-2}}{(2k+c)^2} = \frac{b_{2k-4}}{(2k+c)^2(2k+c-2)^2}$$
$$= \cdots = \frac{b_2}{(2k+c)^2(2k+c-2)^2 \cdots (4+c)^2}$$

for all $k \geqq 2$. Thus we have

$$b_{2k} = \frac{1}{(2+c)^2(4+c)^2 \cdots (2k-2+c)^2(2k+c)^2},$$

a formula valid for all integral $k \geqq 1$.

Reviewing what we have done so far, we find that we have shown that the expression

$$(44) \qquad w_c = z^c + \sum_{k=1}^{\infty} \frac{z^{2k+c}}{[\,(2+c)(4+c)\cdots(2k+c)\,]^2}$$

reduces the left member of equation (42) to c^2z^{c-1}. That is, for the w_c of (44) we have

$$(45) \qquad z\frac{d^2w_c}{dz^2} + \frac{dw_c}{dz} - zw_c = c^2z^{c-1},$$

no matter what c is. Naturally, (44) will satisfy equation (42) if, and only if, $c = 0$. But, when an expression contains as a factor the square of a parameter, not only the original expression but also its partial derivative with respect to that parameter will vanish when the parameter is put equal to zero. In other words, from equation (45) we may conclude at once that

$$z\frac{d^2}{dz^2}\left(\frac{\partial w_c}{\partial c}\right) + \frac{d}{dz}\left(\frac{\partial w_c}{\partial c}\right) - z\left(\frac{\partial w_c}{\partial c}\right) = \frac{\partial}{\partial c}\,(c^2z^{c-1})$$
$$= cz^{c-1}\,[2 + c\log z],$$

and therefore that $\left(\frac{\partial w_c}{\partial c}\right)_{c=0}$ is also a solution of our differential equation. Now, from (44), we find that

$$\frac{\partial w_c}{\partial c} = \left\{z^c + \sum_{k=1}^{\infty} \frac{z^{2k+c}}{[\,(2+c)(4+c)\cdots(2k+c)\,]^2}\right\}\log z$$
$$- \sum_{k=1}^{\infty} \frac{z^{2k+c}\left[\dfrac{2}{2+c} + \dfrac{2}{4+c} + \cdots + \dfrac{2}{2k+c}\right]}{[\,(2+c)(4+c)\cdots(2k+c)\,]^2},$$

and

$$\left(\frac{\partial w_c}{\partial c}\right)_{c=0} = \left\{1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{[2 \cdot 4 \cdots (2k)]^2}\right\} \log z - \sum_{k=1}^{\infty} \frac{z^{2k}\left[\frac{2}{2} + \frac{2}{4} + \cdots + \frac{2}{2k}\right]}{[2 \cdot 4 \cdots (2k)]^2}.$$

Further simplification of the formula yields

$$w_2(z) = \left(\frac{\partial w_c}{\partial c}\right)_{c=0} = \left\{\sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(k!)^2}\right\} \log z - \sum_{k=1}^{\infty} \frac{z^{2k}\left[1 + \frac{1}{2} + \cdots + \frac{1}{k}\right]}{2^{2k}(k!)^2}.$$

Finally we may write

$$w_2(z) = w_1(z) \log z - \sum_{k=1}^{\infty} \frac{H_k z^{2k}}{2^{2k}(k!)^2},$$

where

$$H_k = \sum_{i=1}^{k} \frac{1}{i} \tag{46}$$

is the familiar sum of the first k terms of the harmonic series.

Having obtained w_1 and w_2, evidently linearly independent, we have completed the solution of equation (42). Next a word on a notation frequently found in the literature on Bessel functions. Two linearly independent solutions of

$$z \frac{d^2w}{dz^2} + \frac{dw}{dz} - zw = 0, \tag{42}$$

namely $I_0(z)$ and $K_0(z)$, are called the modified Bessel functions of index zero, $I_0(z)$ being "of the first kind" and $K_0(z)$ being "of the second kind." These must, of course, be linearly dependent upon w_1 and w_2. In fact,

$$I_0(z) = w_1(z),$$
$$K_0(z) = (\log 2 - \gamma)w_1(z) - w_2(z),$$

where

$$\gamma = \lim_{n \to \infty} (H_n - \log n) = 0.577\ 215\ 664\ 9 \cdots$$

is called Euler's or Mascheroni's constant. The logarithms involved are all to the base e, and the H_n in the definition of γ is as defined in equation (46), page 100.

We close this section by formulating a rule for the case $s = 0$. Put

$$w_c = (z - z_0)^c + \sum_{n=1}^{\infty} a_n (z - z_0)^{n+c}.$$

First disregard the indicial equation, except to note that it does have equal roots, but determine the a_n so as to satisfy all the recurrence relations. The a_n then will involve the parameter c. Next obtain the double root $c = c_1 = c_2$ of the indicial equation. Then two linearly independent solutions of the equation are

$$w_1(z) = (w_c)_{c=c_1}$$

and

$$w_2(z) = \left(\frac{\partial w_c}{\partial c}\right)_{c=c_1}.$$

Hence the desired general solution is

$$w = Aw_1(z) + Bw_2(z),$$

where A and B are arbitrary constants. Of course, this rule is formulated only for the case where z_0 is a regular singular point of a second-order linear differential equation and where $s = 0$.

PROBLEMS

1. Solve the equation

$$z^2 \frac{d^2w}{dz^2} + (z^3 - z) \frac{dw}{dz} + (1 - z^2)w = 0.$$

$$\textit{Ans.} \quad w = Az + B\left[z \log z + \sum_{k=1}^{\infty} (-1)^k \frac{z^{2k+1}}{2^{k+1}k(k!)}\right].$$

Hint: $w_c = z^c + \sum_{k=1}^{\infty} (-1)^k \dfrac{(c-1)z^{2k+c}}{(2k+c-1)[(1+c)(3+c)\cdots(2k-1+c)]}$.

2. Solve the equation

$$z^2 \frac{d^2w}{dz^2} - (z + z^2) \frac{dw}{dz} + (1 - z)w = 0.$$

$$\textit{Ans.} \quad w = A \sum_{n=0}^{\infty} \frac{n+1}{n!} z^{n+1}$$
$$+B\left[\log z \sum_{n=0}^{\infty} \frac{n+1}{n!} z^{n+1} - \sum_{n=1}^{\infty} \frac{n + (n+1)H_n}{n!} z^{n+1}\right].$$

3. Obtain solutions valid near $z = 1$.

$$(z-1)\frac{d^2w}{dz^2} + z\frac{dw}{dz} + w = 0.$$

4. Solve the equation

$$z\frac{d^2w}{dz^2} + \frac{dw}{dz} - w = 0.$$

Ans. $w_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^2}$.

$$w_2(z) = w_1(z)\log z - \sum_{n=1}^{\infty} \frac{2H_n}{(n!)^2} z^n.$$

5. Solve the equation

$$4z^2(z+1)\frac{d^2w}{dz^2} + w = 0$$

in the neighborhood of $z = 0$. As a check, note that

$$w_1(z) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{[(2n)!]^2}{2^{4n}(n!)^4(2n-1)} z^{n+(1/2)}.$$

33. Numerical Example: Case III. ***s* a positive integer, nonlogarithmic case.** We shall now treat the case when s is a positive integer and the relation which would ordinarily determine a_s is automatically satisfied. We borrow a numerical example from the problems in Hutchinson's *Elementary Differential Equations*, 1934, 52. Consider

$$z\frac{d^2w}{dz^2} + 2\frac{dw}{dz} + z^2w = 0.$$

This equation has only one singular point in the finite part of the plane, a regular singular point at $z = 0$. . Since $P(z) = 2$ and $Q(z) = z^3$, we have $p_0 = 2$, $q_0 = 0$, and the indicial equation $c^2 + c = 0$. Then $c_1 = 0, c_2 = -1$, and $s = 1$. As was indicated in Section 30, we find it advantageous to use $c = c_2$. Let us put

$$w = \sum_{n=0}^{\infty} a_n z^{n-1}.$$

Then we find from the differential equation that

$$\sum_{n=0}^{\infty} (n-1)(n-2)a_n z^{n-2} + \sum_{n=0}^{\infty} 2(n-1)a_n z^{n-2} + \sum_{n=0}^{\infty} a_n z^{n+1} = 0.$$

Combining the first two series and shifting index in the last yields

$$\sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=3}^{\infty} a_{n-3} z^{n-2} = 0.$$

Let us combine these two series and write

$$2a_2 + \sum_{n=3}^{\infty} [n(n-1)a_n + a_{n-3}]z^{n-2} = 0.$$

Now we see that we must put $a_2 = 0$ and, for $n \geqq 3$,

$$a_n = -\frac{a_{n-3}}{n(n-1)}.$$

Here a_0 and a_1 are arbitrary. That a_0 is arbitrary is the expected result of our using a root of the indicial equation. The relation which might be expected to determine a_s, in our case a_1, is automatically satisfied, leaving that coefficient also arbitrary. This is the distinguishing mark of Case III as opposed to Case IV of the next section. See also the remarks on page 91, near the end of Section 30.

We proceed to complete the solution of our present problem. We now separate the a_n into three sets. Let k be a positive integer. Then from $a_2 = 0$ it follows that $a_{3k+2} = 0$. From the recurrence relation above we find that, for $k \geqq 1$,

$$a_{3k+1} = \frac{-a_{3k-2}}{(3k+1)(3k)} = \frac{(-1)^2 a_{3k-5}}{[(3k+1)(3k-2)][(3k)(3k-3)]} = \cdots$$

$$= \frac{(-1)^k a_1}{[(3k+1)(3k-2)\cdots 4][(3k)(3k-3)\cdots 3]},$$

and

$$a_{3k} = \frac{-a_{3k-3}}{(3k)(3k-1)} = \frac{(-1)^2 a_{3k-6}}{[(3k)(3k-3)][(3k-1)(3k-4)]} = \cdots$$

$$= \frac{(-1)^k a_0}{[(3k)(3k-3)\cdots 3][(3k-1)(3k-4)\cdots 2]}.$$

Next we introduce appropriate factors in numerator and denominator in order to express the denominator more simply as a factorial in each case. This leads us, for $k \geqq 1$, to the values

$$a_{3k+1} = (-1)^k \frac{2 \cdot 5 \cdot 8 \cdots (3k-1)}{(3k+1)!} a_1,$$

$$a_{3k} = (-1)^k \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k)!} a_0.$$

Our expression for w has become

$$\begin{aligned} w &= a_0 \left[z^{-1} + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{(3k)!} z^{3k-1} \right] \\ &\qquad + a_1 \left[1 + \sum_{k=1}^{\infty} (-1)^k \frac{2 \cdot 5 \cdot 8 \cdots (3k-1)}{(3k+1)!} z^{3k} \right] \\ &= a_0 w_2(z) + a_1 w_1(z), \end{aligned}$$

which is the desired complete solution. Note, moreover, that the coefficient of a_1 is precisely the solution we would have obtained had we started out using $c = c_1 = 0$.

PROBLEMS

1. Solve the equation

$$z^2(1+z)\frac{d^2w}{dz^2} + z(1+z)\frac{dw}{dz} - w = 0.$$

$$\textit{Ans.}\quad w_1(z) = \sum_{n=0}^{\infty} (-1)^n \frac{2}{(n+1)(n+2)} z^{n+1}.$$

$$w_2(z) = z^{-1} + 1.$$

2. Solve the equation

$$z\frac{d^2w}{dz^2} + (z-1)\frac{dw}{dz} - w = 0.$$

$$\begin{aligned} \textit{Ans.}\quad w_1(z) &= \sum_{n=0}^{\infty} (-1)^n \frac{2}{(n+2)!} z^{n+2} \\ &= 2(e^{-z} - 1 + z). \\ w_2(z) &= 1 - z. \\ w(z) &= A(1-z) + Be^{-z}. \end{aligned}$$

3. Solve the equation

$$z\frac{d^2w}{dz^2} + (z^3 - 1)\frac{dw}{dz} + 2z^2w = 0.$$

$$\textit{Ans.}\quad w_1(z) = \sum_{k=0}^{\infty} (-1)^k \frac{2[1\cdot 4\cdot 7\cdots (3k+1)]^2}{(3k+2)!} z^{3k+2}.$$

$$w_2(z) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{[2\cdot 5\cdot 8\cdots (3k-1)]^2}{(3k)!} z^{3k}.$$

4. Solve the following equation by the methods of the present section, and then check your result by solving the equation with the aid of the normal form, Section 3.

$$4z^2\frac{d^2w}{dz^2} + 4z(2z - 1)\frac{dw}{dz} + (3 - 4z)w = 0.$$

5. Solve the equation

$$9z^2\frac{d^2w}{dz^2} + 3z(z^2 + 2)\frac{dw}{dz} + (z^2 - 2)w = 0.$$

$$\textit{Ans.}\quad w_1(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+(2/3)}}{6^k k!(2k+1)}.$$

$$w_2(z) = z^{-1/3}.$$

6. Solve the equation

$$z^2(1 - z^4)\frac{d^2w}{dz^2} - 2w = 0.$$

7. Solve the equation

$$z^2(z + 4)\frac{d^2w}{dz^2} - 4z\frac{dw}{dz} + 4w = 0.$$

8. Solve the equation

$$z^2(1 + z^7)\frac{d^2w}{dz^2} - 6w = 0.$$

34. Numerical Example: Case IV. *s* a positive integer, logarithmic case. We now consider an example in which s is a positive integer but, as opposed to Case III, the relation which would normally determine a_s turns out to be impossible. We shall use the ordinary Bessel equation of index 2.

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - 4)w = 0. \tag{47}$$

At once we see that $P(z) = 1$ and $Q(z) = z^2 - 4$. From these

$p_0 = 1$ and $q_0 = -4$, so that our indicial equation becomes $c^2 - 4 = 0$. Thus $c_1 = 2$, $c_2 = -2$, and $s = 4$.

Once more, since in practice we would not know whether this equation belonged to Case III or Case IV, we use $c = c_2 = -2$. We put

$$w = \sum_{n=0}^{\infty} a_n z^{n-2}$$

and find at once

$$\sum_{n=0}^{\infty} (n-2)(n-3)a_n z^{n-2} + \sum_{n=0}^{\infty} (n-2)a_n z^{n-2} + \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} 4a_n z^{n-2} = 0.$$

Combining terms and shifting the index in the next to last summation, we find

$$\sum_{n=0}^{\infty} [(n-2)(n-3) + (n-2) - 4]a_n z^{n-2} + \sum_{n=2}^{\infty} a_{n-2} z^{n-2} = 0,$$

or

$$\sum_{n=0}^{\infty} n(n-4)a_n z^{n-2} + \sum_{n=2}^{\infty} a_{n-2} z^{n-2} = 0.$$

Thus we have

$$-3a_1 z^{-1} + \sum_{n=2}^{\infty} [n(n-4)a_n + a_{n-2}]z^{n-2} = 0.$$

From this identity we have at once the equations

$$\begin{aligned} a_1 &= 0, \\ -4a_2 + a_0 &= 0, \\ -3a_3 + a_1 &= 0, \\ a_2 &= 0, \end{aligned}$$

and, for $n \geqq 5$,

$$a_n = -\frac{a_{n-2}}{n(n-4)}.$$

The equation $a_2 = 0$ came from the relation which was supposed to have determined $a_4 = a_s$. Here we see how Case IV differs from Case III where the corresponding equation was automatically satisfied, not imposing any condition such as $a_2 = 0$. From the second of the above relations we may conclude that $a_0 = 0$, which is impossible, if we adhere to our customary assumption $a_0 \neq 0$. If we were to remove that assumption, we would merely obtain the solution $w_1(z)$ corresponding to the exponent $c_1 = +2$. It seems more reasonable to return to that exponent to obtain the solution belonging to it.

Therefore we take $c = c_1 = 2$ and put

$$w = \sum_{n=0}^{\infty} b_n z^{n+2}.$$

Now we have from

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - 4)w = 0 \tag{47}$$

the identity

$$\sum_{n=0}^{\infty} (n+2)(n+1)b_n z^{n+2} + \sum_{n=0}^{\infty} (n+2)b_n z^{n+2} + \sum_{n=0}^{\infty} b_n z^{n+4} - \sum_{n=0}^{\infty} 4b_n z^{n+2} = 0.$$

From the above identity we deduce that

$$\sum_{n=0}^{\infty} n(n+4)b_n z^{n+2} + \sum_{n=2}^{\infty} b_{n-2} z^{n+2} = 0,$$

or

$$5b_1 z^3 + \sum_{n=2}^{\infty} [n(n+4)b_n + b_{n-2}]z^{n+2} = 0.$$

We are thus led to the equations $b_1 = 0$, and, for $n \geqq 2$,

$$b_n = -\frac{b_{n-2}}{n(n+4)}.$$

Then, for integral $k \geqq 0$, $b_{2k+1} = 0$ follows from $b_1 = 0$ and the recurrence relation. For even $n = 2k$, we have, for $k \geqq 1$,

$$b_{2k} = \frac{-b_{2k-2}}{(2k+4)(2k)} = \frac{(-1)^2 b_{2k-4}}{(2k+4)(2k)(2k+2)(2k-2)}$$
$$= \frac{(-1)^3 b_{2k-6}}{(2k+4)(2k)(2k+2)(2k-2)(2k)(2k-4)}.$$

This relation is clearer when written in the form

$$b_{2k} = \frac{(-1)^3 b_{2k-6}}{[(2k+4)(2k+2)(2k)][(2k)(2k-2)(2k-4)]}.$$

Continued reduction of the subscript on the right leads us to

$$b_{2k} = \frac{(-1)^k b_0}{[(2k+4)(2k+2)\cdots 10\cdot 8\cdot 6][(2k)(2k-2)\cdots 6\cdot 4\cdot 2]}.$$

We may now factor 2 out of each factor in the denominator of b_{2k} and write

$$\begin{aligned} b_{2k} &= \frac{(-1)^k b_0}{2^{2k}[(k+2)(k+1)\cdots 5\cdot 4\cdot 3][k(k-1)\cdots 3\cdot 2\cdot 1]} \\ &= \frac{(-1)^k b_0}{2^{2k-1}(k+2)!k!}. \end{aligned}$$

Thus we find that our solution is

$$\begin{aligned} w &= b_0\left[z^2 + \sum_{k=1}^{\infty} (-1)^k \frac{z^{2k+2}}{2^{2k-1}(k+2)!k!}\right] \\ &= b_0 \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+2}}{2^{2k-1}(k+2)!k!} = b_0 w_1(z). \end{aligned}$$

The classical notation for the Bessel function of the first kind and of index 2 is

$$J_2(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+2}}{2^{2k+2}(k+2)!k!}.$$

Thus in standard notation we may write $w_1(z) = 8J_2(z)$.

It is now our task to obtain another solution of equation (47). It is possible to proceed as we did in Case II, Section 32, with slight variations. We shall delay the use of that method until Section 36.

A peculiar variation of the usual methods is to be found in Forsyth (Articles 104 and 105 of the sixth edition). Forsyth obtains a second solution in the form $J_2(z) \log z$ plus an infinite series each term of which contains a Bessel function of the first kind with an integral index which varies with the term. His work is actually performed on the Bessel equation of order n, whereas ours is for the special case $n = 2$.

We shall take this opportunity to introduce another method for finding a second solution in the troublesome Cases II and IV. The numerical example is continued into the next section.

35. Continuation of the Example for Case IV. We may profit by the experience gained in Case II and attempt to determine a second solution to the equation

$$z^2 \frac{d^2w}{dz^2} + z \frac{dw}{dz} + (z^2 - 4)w = 0 \tag{47}$$

by introducing a logarithm into our assumed form for $w_2(z)$. Let us take

$$w_2(z) = w_1 \log z + w_3,$$

where we hope to be able to determine $w_3(z)$ as a series involving powers of z. Now

$$\frac{dw_2}{dz} = \frac{dw_1}{dz} \log z + z^{-1}w_1 + \frac{dw_3}{dz},$$

$$\frac{d^2w_2}{dz^2} = \frac{d^2w_1}{dz^2} \log z + 2z^{-1} \frac{dw_1}{dz} - z^{-2}w_1 + \frac{d^2w_3}{dz^2},$$

so that substitution into equation (47) yields

$$\left[z^2 \frac{d^2w_1}{dz^2} + z \frac{dw_1}{dz} + (z^2 - 4)w_1\right] \log z + 2z \frac{dw_1}{dz} - w_1 + w_1$$
$$+ z^2 \frac{d^2w_3}{dz^2} + z \frac{dw_3}{dz} + (z^2 - 4)w_3 = 0.$$

Since w_1 is a solution of equation (47), the terms involving $\log z$ drop out, and we have

$$z^2 \frac{d^2w_3}{dz^2} + z \frac{dw_3}{dz} + (z^2 - 4)w_3 + 2z \frac{dw_1}{dz} = 0. \tag{48}$$

We shall assume

$$w_3(z) = \sum_{n=0}^{\infty} d_n z^{n+\sigma},$$

where we may take $d_0 \neq 0$ and attempt to determine σ and the d_n from equation (48). Intuitively we may expect σ to be a solution of the indicial equation of the original problem, indeed, to be equal to c_2.

Since

$$w_1 = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+2}}{2^{2k-1}(k+2)!k!},$$

we see that we must have

$$\sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-1)d_n z^{n+\sigma} + \sum_{n=0}^{\infty} (n+\sigma)d_n z^{n+\sigma} + \sum_{n=0}^{\infty} d_n z^{n+\sigma+2} - \sum_{n=0}^{\infty} 4d_n z^{n+\sigma} + \sum_{k=0}^{\infty} (-1)^k \frac{2(2k+2)z^{2k+2}}{2^{2k-1}(k+2)!k!} = 0.$$

Next we may write

$$\sum_{n=0}^{\infty} [(n+\sigma)^2 - 4]d_n z^{n+\sigma} + \sum_{n=2}^{\infty} d_{n-2} z^{n+\sigma} + \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)z^{2k+2}}{2^{2k-3}(k+2)!k!} = 0,$$

or

$$(\sigma^2 - 4)d_0 z^\sigma + [(\sigma+1)^2 - 4]d_1 z^{1+\sigma} + \sum_{n=2}^{\infty} \left\{[(n+\sigma)^2 - 4]d_n + d_{n-2}\right\} z^{n+\sigma} + \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)z^{2k+2}}{2^{2k-3}(k+2)!k!} = 0.$$

In the above identity the first term of the series which came from $w_1(z)$ is $4z^2$. Now either the term $(\sigma^2 - 4)d_0 z^\sigma$ must cancel this $4z^2$ or the term $(\sigma^2 - 4)d_0 z^\sigma$ must vanish and some later term in the series must cancel the $4z^2$. That is, either

(*a*) $\sigma = 2$ and $(\sigma^2 - 4)d_0 + 4 = 0$,

or

(*b*) $\sigma < 2$ and $(\sigma^2 - 4)d_0 = 0$.

But (*a*) is impossible. Since we have chosen $d_0 \neq 0$, we may conclude from (*b*) that $\sigma^2 = 4$ while $\sigma < 2$. Thus we must take $\sigma = -2$, which is our original c_2.

Using $\sigma = -2$, we may write our identity as

$$-3d_1z^{-1} + \sum_{n=2}^{\infty} [n(n-4)d_n + d_{n-2}]z^{n-2} + \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)z^{2k+2}}{2^{2k-3}(k+2)!k!} = 0.$$

Once more we need to separate our treatment into the two cases, n even and n odd. Since there is only one term involving z^{-1}, we have $d_1 = 0$. For $n = 2k+1$, k integral,

$$(2k+1)(2k-3)d_{2k+1} = -d_{2k-1},$$

so that $d_{2k+1} = 0$ follows from $d_1 = 0$. Now we may put $n = 2k$, shift the index in the latter summation, and write

$$\sum_{k=1}^{\infty} [2k(2k-4)d_{2k} + d_{2k-2}]z^{2k-2} + \sum_{k=2}^{\infty} (-1)^k \frac{(k-1)z^{2k-2}}{2^{2k-7}k!(k-2)!} = 0,$$

where we have replaced $(-1)^{k-2}$ by its equivalent $(-1)^k$.

The equations for the determination of d_{2k} are now seen to be:

for $k = 1$: $\quad -4d_2 + d_0 = 0$

for $k = 2$: $\quad 0 \cdot d_4 + d_2 + 4 = 0$

for $k \geqq 3$:

$$2^2k(k-2)d_{2k} + d_{2k-2} + (-1)^k \frac{k-1}{2^{2k-7}k!(k-2)!} = 0.$$

Of course, we have $d_2 = -4$, $d_0 = -16$. Further, it is easy to see that d_4 is arbitrary, since the relation for $k = 3$ may be used to determine d_6 in terms of d_4. We shall choose d_4 later. At the moment we wish to simplify the general recurrence formula for d_{2k}. With this in mind we put, for $k \geqq 2$,

$$d_{2k} = (-1)^k \frac{f_{2k}}{2^{2k-5}k!(k-2)!},$$

where the f_{2k} are now to be determined. From this we have

$$d_{2k-2} = (-1)^{k-1} \frac{f_{2k-2}}{2^{2k-7}(k-1)!(k-3)!},$$

and the recurrence relation for the f_{2k} appears as

$$(-1)^k \frac{k(k-2)}{2^{2k-7}k!(k-2)!} f_{2k} + (-1)^{k-1} \frac{f_{2k-2}}{2^{2k-7}(k-1)!(k-3)!} + (-1)^k \frac{k-1}{2^{2k-7}k!(k-2)!} = 0.$$

We clear of fractions and have

$$k(k-2)f_{2k} - k(k-2)f_{2k-2} + k - 1 = 0,$$

or

$$f_{2k} = f_{2k-2} - \frac{k-1}{k(k-2)}.$$

This relation, for $k \geqq 3$, is to determine the f_{2k} in terms of f_4, which is arbitrary since d_4 was arbitrary. Next we solve the recurrence relation. Note that

$$\frac{k-1}{k(k-2)} = \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k-2}\right).$$

Then

$$\begin{aligned} f_{2k} &= f_{2k-2} - \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k-2}\right) \\ &= f_{2k-4} - \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k-2} + \frac{1}{k-1} + \frac{1}{k-3}\right) \\ &= f_{2k-6} - \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k-2} + \frac{1}{k-1} + \frac{1}{k-3} + \frac{1}{k-2} + \frac{1}{k-4}\right). \end{aligned}$$

The last of these expressions for f_{2k} is more conveniently written

$$f_{2k} = f_{2k-6} - \frac{1}{2}\left[\left(\frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2}\right) + \left(\frac{1}{k-2} + \frac{1}{k-3} + \frac{1}{k-4}\right)\right],$$

grouping alternate terms in the previous form. We proceed to reduce the subscript on the right, noting that the final substitutions use

$$f_8 = f_6 - \tfrac{1}{2}(\tfrac{1}{4} + \tfrac{1}{2}),$$

and

$$f_6 = f_4 - \tfrac{1}{2}(\tfrac{1}{3} + \tfrac{1}{1}).$$

Thus we have, for $k \geqq 3$,

$$f_{2k} = f_4 - \frac{1}{2}\left[\left(\frac{1}{k} + \frac{1}{k-1} + \frac{1}{k-2} + \cdots + \frac{1}{4} + \frac{1}{3}\right)\right.$$
$$\left. + \left(\frac{1}{k-2} + \frac{1}{k-3} + \frac{1}{k-4} + \cdots + \frac{1}{2} + 1\right)\right],$$

or

$$f_{2k} = f_4 - \tfrac{1}{2}(H_k + H_{k-2} - \tfrac{1}{2} - 1),$$

in terms of the H_k of

(46) $$H_k = \sum_{i=1}^{k} \frac{1}{i}.$$

Naturally we choose $f_4 = -\frac{1}{2}(\frac{1}{2} + 1)$. Finally we have

$$f_{2k} = -\tfrac{1}{2}(H_k + H_{k-2}),$$

so that, for $k \geqq 3$,

$$d_{2k} = \frac{(-1)^{k+1}(H_k + H_{k-2})}{2^{2k-4}k!(k-2)!}.$$

Now

$$d_4 = (-1)^2 \frac{f_4}{2^{-1}2!0!} = \frac{-\frac{1}{2}(\frac{1}{2}+1)}{1} = -\frac{H_2}{2}.$$

Therefore, if we define $H_0 = 0$, we may write

$$d_4 = -\frac{H_2 + H_0}{2} = \frac{(-1)^{2+1}(H_2 + H_0)}{2^{4-4}2!0!},$$

which fits into the general formula for d_{2k}.

Finally we may write $w_3(z)$ as

$$w_3(z) = -16z^{-2} - 4 + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{H_k + H_{k-2}}{2^{2k-4}k!(k-2)!} z^{2k-2},$$

or

$$w_3(z) = -16z^{-2} - 4 + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{H_{k+2} + H_k}{2^{2k}(k+2)!k!} z^{2k+2},$$

more nearly resembling the expression for

$$w_1(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+2}}{2^{2k-1}k!(k+2)!}.$$

The complete solution of the equation

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-4)w=0 \tag{47}$$

is now seen to be

$$w=Aw_1(z)+B\,[w_1(z)\log z+w_3(z)],$$

where A and B are arbitrary constants.

We have already remarked that $w_1(z)=8J_2(z)$, where $J_2(z)$ is that particular solution of equation (47) which is usually called the Bessel function of the first kind of index 2. Naturally, $w_2(z)$ would be referred to as a Bessel function of the second kind of index 2. Since we are not primarily interested in Bessel functions, we shall not try to relate $w_2(z)$ to any of the unfortunately numerous* notations for Bessel functions of the second kind.

PROBLEMS

1. Solve the equation

$$z\frac{d^2w}{dz^2}-2\frac{dw}{dz}+2w=0.$$

Ans. $w=Aw_1(z)+Bw_2(z)$, where

$$w_1(z)=\sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 2^{n+1}}{(n+3)!n!}z^{n+3},$$

$$w_2(z)=w_1(z)\log z-\tfrac{3}{2}(1+z+z^2)-\sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 2^{n+1}(H_{n+3}+H_n)}{(n+3)!n!}z^{n+3}.$$

2. Solve the following equation by obtaining $w_1(z)$ and then using the method of Section 2 to complete the solution. Resolve the equation by the method of the present section.

$$z(1+z)\frac{d^2w}{dz^2}-\frac{dw}{dz}-2w=0.$$

Ans. $w_1(z)=z^2.$

$$\begin{aligned}w_2(z)&=z^2\log z-\tfrac{1}{2}+z+\sum_{n=3}^{\infty}(-1)^n\frac{z^n}{n-2}\\&=z^2\log z-\tfrac{1}{2}+z+\sum_{n=1}^{\infty}(-1)^n\frac{z^{n+2}}{n}\\&=z^2\,[\log z-\log(1+z)]-\tfrac{1}{2}+z.\end{aligned}$$

---

* See Whittaker and Watson, *op. cit.*, 372, for three of these.

3. Solve the equation $$z\frac{d^2w}{dz^2} - w = 0.$$

Ans. $$w_1(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!(n+1)!}.$$

$$w_2(z) = w_1(z)\log z + 1 - z - \sum_{n=2}^{\infty} \frac{H_n + H_{n-1}}{n!(n-1)!} z^n$$

$$= w_1(z)\log z + 1 - \sum_{n=0}^{\infty} \frac{H_{n+1} + H_n}{n!(n+1)!} z^{n+1}.$$

4. Solve the equation

$$z\frac{d^2w}{dz^2} + (z^2 - 1)\frac{dw}{dz} + zw = 0.$$

36. An Alternative Method for Case IV. Let us now turn to another method for obtaining a second solution in Case IV. This method is closely allied to the one we followed in treating Case II; it may be used in place of the method of Section 35.

We proceed on the assumption that we have gone through the work of Section 34, but not that of Section 35. We put

$$w = \sum_{n=0}^{\infty} g_n z^{n+c}$$

and find that

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - 4)w$$

$$= \sum_{n=0}^{\infty} (n+c)(n+c-1)g_n z^{n+c} + \sum_{n=0}^{\infty} (n+c)g_n z^{n+c} + \sum_{n=0}^{\infty} g_n z^{n+c+2} - \sum_{n=0}^{\infty} 4g_n z^{n+c}$$

$$= \sum_{n=0}^{\infty} [(n+c)^2 - 4]g_n z^{n+c} + \sum_{n=2}^{\infty} g_{n-2} z^{n+c}$$

$$= (c^2 - 4)g_0 z^c + [(1+c)^2 - 4]g_1 z^{1+c} + \sum_{n=2}^{\infty} [(n+c+2)(n+c-2)g_n + g_{n-2}]z^{n+c}.$$

We are able, without choosing c or g_0, to make all except the first term in this last expression vanish. In order to do this we put $g_1 = 0$ and, for $n \geqq 2$,

$$g_n = \frac{-g_{n-2}}{(n+c+2)(n+c-2)}.$$

Treating this recurrence relation in a manner which is by this time familiar to the student, we find that, for $k \geqq 1$, $g_{2k-1} = 0$, and

$$g_{2k} = \frac{(-1)^k g_0}{A_{2k}\,[(2k + c - 2)(2k + c - 4)\cdots(2 + c)(c)]},$$

where

$$A_{2k} = (2k + c + 2)(2k + c)\cdots(6 + c)(4 + c).$$

The A_{2k} has been introduced solely to avoid awkwardness in printing. It would be advisable for the student to insert for A_{2k} throughout this section the explicit expression in terms of k and c.

Now we know that for the function

$$w = g_0 z^c + \sum_{k=1}^{\infty} (-1)^k \frac{g_0 z^{2k+c}}{A_{2k}\,[(2k + c - 2)(2k + c - 4)\cdots(2 + c)(c)]}$$

we have

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - 4)w = (c^2 - 4)g_0 z^c. \tag{49}$$

At this stage we digress from the method of Section 32. It is evident that we cannot now choose $c = -2$, since the factor $(c + 2)$ occurs in the denominator of some terms of the series for w. Let us, then, replace the arbitrary constant g_0 by $(c + 2)h$. This will eliminate the troublesome factor $(c + 2)$ from the denominators of our series. Further, equation (49) in terms of the new w, which we shall call w_c, becomes

$$z^2\frac{d^2w_c}{dz^2} + z\frac{dw_c}{dz} + (z^2 - 4)w_c = (c + 2)^2(c - 2)hz^c.$$

The right member of the above equation has $c = -2$ as a double zero. Therefore both $(w_c)_{c=-2}$ and $\left(\frac{\partial w_c}{\partial c}\right)_{c=-2}$ are solutions of our original differential equation,

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - 4)w = 0. \tag{47}$$

But $(w_c)_{c=-2} = -\dfrac{w_1(z)}{16}$, as may be seen from equation (50) below. Hence we proceed toward the new solution

$$w_2 = \left(\frac{\partial w_c}{\partial c}\right)_{c=-2}.$$

If we write

$$w_c = (c+2)hz^c + \sum_{k=1}^{\infty}(-1)^k \frac{(c+2)hz^{2k+c}}{A_{2k}\,[(c)(2+c)\cdots(2k-2+c)]},$$

in the form

$$w_c = (c+2)hz^c - \frac{c+2}{(4+c)(c)}hz^{2+c} + \frac{c+2}{(4+c)(6+c)(c)(2+c)}hz^{4+c}$$
$$+ \sum_{k=3}^{\infty}(-1)^k \frac{(c+2)hz^{2k+c}}{A_{2k}\,[(c)(2+c)\cdots(2k-2+c)]},$$

it is easier to see just where we may cancel the $(c+2)$ in numerator and denominator. Finally we write

$$w_c = (c+2)hz^c - \frac{c+2}{(4+c)(c)}hz^{2+c} + \frac{1}{(4+c)(6+c)(c)}hz^{4+c}$$

(50)

$$+ \sum_{k=3}^{\infty}(-1)^k \frac{hz^{2k+c}}{A_{2k}\,[(4+c)(6+c)\cdots(2k-2+c)](c)}.$$

Now we are in a position to obtain w_2. First we have

$$\frac{\partial w_c}{\partial c} = w_c \log z + hz^c - \frac{hz^{2+c}}{(4+c)(c)} + \frac{(c+2)hz^{2+c}}{(4+c)(c)}\left(\frac{1}{4+c}+\frac{1}{c}\right)$$
$$- \frac{hz^{4+c}}{(4+c)(6+c)(c)}\left(\frac{1}{4+c}+\frac{1}{6+c}+\frac{1}{c}\right)$$
$$+ \sum_{k=3}^{\infty}(-1)^{k+1} \frac{hz^{2k+c}\left(\dfrac{1}{4+c}+\dfrac{1}{6+c}+\cdots+\dfrac{1}{2k+2+c}\right)}{A_{2k}\,[(4+c)(6+c)\cdots(2k-2+c)](c)}$$
$$+ \sum_{k=3}^{\infty}(-1)^{k+1} \frac{hz^{2k+c}\left(\dfrac{1}{4+c}+\dfrac{1}{6+c}+\cdots+\dfrac{1}{2k-2+c}+\dfrac{1}{c}\right)}{A_{2k}\,[(4+c)(6+c)\cdots(2k-2+c)](c)}.$$

Next we may obtain $w_2 = \left(\dfrac{\partial w_c}{\partial c}\right)_{c=-2}$ in the form

$$
\begin{aligned}
w_2 &= \frac{hz^2 \log z}{(2)(4)(-2)} \\
&\quad + \sum_{k=3}^{\infty} (-1)^k \frac{hz^{2k-2} \log z}{[2 \cdot 4 \cdot 6 \cdots (2k)][2 \cdot 4 \cdot 6 \cdots (2k-4)](-2)} \\
&\quad + hz^{-2} - \frac{h}{(2)(-2)} - \frac{hz^2}{(2)(4)(-2)} \left(\tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{2}\right) \\
&\quad + \sum_{k=3}^{\infty} (-1)^{k+1} \frac{hz^{2k-2}\left(\dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2k}\right)}{[2 \cdot 4 \cdot 6 \cdots (2k)][2 \cdot 4 \cdot 6 \cdots (2k-4)](-2)} \\
&\quad + \sum_{k=3}^{\infty} (-1)^{k+1} \frac{hz^{2k-2}\left(\dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2k-4} - \dfrac{1}{2}\right)}{[2 \cdot 4 \cdot 6 \cdots (2k)][2 \cdot 4 \cdot 6 \cdots (2k-4)](-2)}.
\end{aligned}
$$

With our usual methods of simplification, we find that the above reduces to

$$
\begin{aligned}
w_2 &= h\left[-\frac{1}{2^4} z^2 + \sum_{k=3}^{\infty} (-1)^{k+1} \frac{z^{2k-2}}{2^{2k-1} k!(k-2)!}\right] \log z \\
&\qquad + h\left[z^{-2} + \frac{1}{4} + \frac{z^2}{64} + \sum_{k=3}^{\infty} (-1)^k \frac{(H_k + H_{k-2} - 1)}{2^{2k} k!(k-2)!} z^{2k-2}\right],
\end{aligned}
$$

or

$$
\begin{aligned}
w_2 &= h \sum_{k=2}^{\infty} (-1)^{k+1} \frac{z^{2k-2}}{2^{2k-1} k!(k-2)!} \log z \\
&\qquad + h\left[z^{-2} + \frac{1}{4} + \sum_{k=2}^{\infty} (-1)^k \frac{(H_k + H_{k-1} - 1)}{2^{2k} k!(k-2)!} z^{2k-2}\right].
\end{aligned}
$$

Let us alter the form of w_2 to one more easily compared with w_1. We may write

$$
\begin{aligned}
w_2 &= h \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k+2} \log z}{2^{2k+3}(k+2)!k!} + h \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k+2}}{2^{2k+4}(k+2)!k!} \\
&\qquad + h\left[z^{-2} + \frac{1}{4} + \sum_{k=0}^{\infty} (-1)^k \frac{H_{k+2} + H_k}{2^{2k+4}(k+2)!k!} z^{2k+2}\right].
\end{aligned}
$$

Now we may use the definition of $w_1(z)$ to enable us to write

$$w_2 = -\frac{h}{16}w_1(z)\log z - \frac{h}{32}w_1(z)$$
$$-\frac{h}{16}\left[-16z^{-2} - 4 + \sum_{k=0}^{\infty}(-1)^{k+1}\frac{H_{k+2}+H_k}{2^{2k}(k+2)!k!}z^{2k+2}\right].$$

In the form directly above there is a term which is a constant multiple of $w_1(z)$. That term can be absorbed into the term containing $w_1(z)$. Finally, using a new constant $B = -\frac{h}{16}$, we write the complete solution

$$w = Aw_1(z) + B\left[w_1(z)\log z - 16z^{-2} - 4\right.$$
$$\left.+\sum_{k=0}^{\infty}(-1)^{k+1}\frac{H_{k+2}+H_k}{2^{2k}(k+2)!k!}z^{2k+2}\right],$$

as at the end of Section 35.

The student will find numerous opportunities to use the methods for Case IV in the problems which arise in Section 47 and throughout Chapter IX.

PROBLEMS

1. Solve the equation

$$4z^2\frac{d^2w}{dz^2} + 4z\frac{dw}{dz} + (z-1)w = 0.$$

Ans. $w_1(z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{n+(1/2)}}{2^{2n}n!(n+1)!}.$

$$w_2(z) = w_1(z)\log z - 4z^{-1/2} - \sum_{n=0}^{\infty}(-1)^n\frac{H_{n+1}+H_n}{2^{2n}n!(n+1)!}z^{n+(1/2)}.$$

2. Solve the equation

$$z(1+z)\frac{d^2w}{dz^2} - \frac{dw}{dz} - 2w = 0$$

of Problem 2, Section 35, by the method of the present section.

3. Solve the equation

$$z^2\frac{d^2w}{dz^2} + 3z\frac{dw}{dz} - (z+3)w = 0.$$

4. Solve the equation

$$z^2\frac{d^2w}{dz^2} - (6+z^5)w = 0.$$

5. Solve the equation

$$z\frac{d^2w}{dz^2} - w = 0$$

of Problem 3, Section 35, by the method of the present section.

37. Example with a Three-Term Recurrence Relation. Consider the equation

$$z\frac{d^2w}{dz^2} + \frac{dw}{dz} + (1 + z)w = 0.$$

Noting the presence of but one singular point in the finite plane, the regular singular point $z = 0$, we proceed to obtain solutions in the neighborhood of that point. In our usual notation, $P(z) = 1$, $Q(z) = z + z^2$, $p_0 = 1$, $q_0 = 0$, $c_1 = c_2 = 0$, $s = 0$. We put

$$w = \sum_{n=0}^{\infty} a_n z^n,$$

and find

$$\sum_{n=0}^{\infty} n(n-1)a_n z^{n-1} + \sum_{n=0}^{\infty} na_n z^{n-1} + \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+1} = 0.$$

This identity may be rewritten as

$$\sum_{n=0}^{\infty} n^2 a_n z^{n-1} + \sum_{n=1}^{\infty} a_{n-1} z^{n-1} + \sum_{n=2}^{\infty} a_{n-2} z^{n-1} = 0.$$

Finally we write

$$a_1 + a_0 + \sum_{n=2}^{\infty} [n^2 a_n + a_{n-1} + a_{n-2}] z^{n-1} = 0.$$

Of course, a_0 is arbitrary. The other a_n are determined in terms of it by means of the recurrence relations, $a_1 = -a_0$, and, for $n \geqq 2$,

$$a_n = -\frac{a_{n-1} + a_{n-2}}{n^2}.$$

Here we have a relation* from which, in terms of a_0, we may obtain numerically as many of the coefficients as we wish. We cannot, however, find by purely elementary devices an explicit formula for a_n in terms of n. This need not deter us in any sense. We know from the general theory, Section 30, that this power-series solution is convergent for all finite z.

---

* Such a relation is called a finite difference equation. Much study has been made of such equations, and a great deal of literature is devoted to them. That theory bears a marked resemblance to that of differential equations. In a sense, of course, the difference equation we have just encountered characterizes our original differential equation.

Let us choose $a_0 = 1$ so as to be able to write a particular solution

$$w_1(z) = 1 - z + \sum_{n=2}^{\infty} a_n z^n,$$

where, for $n \geqq 2$, a_n is determined by

$$a_n = -\frac{a_{n-1} + a_{n-2}}{n^2}, \tag{51}$$

in which $a_0 = 1$, $a_1 = -1$.

Purely in order to satisfy our curiosity we write out a few terms of the above series

$$\begin{aligned} w_1(z) = 1 - z + \frac{1}{3^2} z^3 - \frac{1}{(3 \cdot 4)^2} z^4 - \frac{4^2 - 1}{(3 \cdot 4 \cdot 5)^2} z^5 \\ + \frac{5^2 + 4^2 - 1}{(3 \cdot 4 \cdot 5 \cdot 6)^2} z^6 + \frac{6^2 \cdot 4^2 - 6^2 - 5^2 - 4^2 + 1}{(3 \cdot 4 \cdot 5 \cdot 6 \cdot 7)^2} z^7 + \cdots \\ + a_n z^n + \cdots, \end{aligned}$$

or

$$\begin{aligned} w_1(z) = 1 - z + \frac{1}{9} z^3 - \frac{1}{144} z^4 - \frac{1}{240} z^5 + \frac{1}{3240} z^6 + \frac{5}{63504} z^7 \\ + \cdots + a_n z^n + \cdots. \end{aligned}$$

It is now our desire to get another solution to the differential equation. The roots of the indicial equation are equal, so we proceed to use either the method of Section 32 or that of Section 35. We shall obtain the desired solution both ways in order to gather more experience.

First let us put

$$w_2(z) = w_1(z) \log z + w_3(z).$$

Then, by direct substitution we find

$$z \frac{d^2 w_3}{dz^2} + \frac{dw_3}{dz} + (1 + z) w_3 + 2 \frac{dw_1}{dz} = 0.$$

If we put

$$w_3 = \sum_{n=0}^{\infty} b_n z^{n+\sigma},$$

then we obtain in the usual manner

$$\sum_{n=0}^{\infty} (n+\sigma)^2 b_n z^{n+\sigma-1} + \sum_{n=1}^{\infty} b_{n-1} z^{n+\sigma-1} + \sum_{n=2}^{\infty} b_{n-2} z^{n+\sigma-1} + \sum_{n=1}^{\infty} 2na_n z^{n-1} = 0,$$

in which the a_n are defined as in (51). As before, we write

$$\sigma^2 b_0 z^{\sigma-1} + [(1+\sigma)^2 b_1 + b_0]z^{\sigma} + \sum_{n=2}^{\infty} [(n+\sigma)^2 b_n + b_{n-1} + b_{n-2}]z^{n+\sigma-1} + \sum_{n=1}^{\infty} 2na_n z^{n-1} = 0.$$

Evidently we may choose $\sigma = 0$, b_0 arbitrarily, and the remaining b_n as determined by the relations

$$b_1 + b_0 + 2a_1 = 0,$$

and, for $n \geqq 2$,

$$n^2 b_n + b_{n-1} + b_{n-2} + 2na_n = 0.$$

Let us take $b_0 = 1$; then we have the solution

$$w_2(z) = w_1(z) \log z + \sum_{n=0}^{\infty} b_n z^n,$$

where the b_n are determined as follows: $b_0 = 1$, $b_1 = 1$, and, for $n \geqq 2$,

$$(52) \qquad b_n = -\frac{b_{n-1} + b_{n-2}}{n^2} - \frac{2a_n}{n},$$

where the a_n are as defined for $w_1(z)$, page 121.

We may write our general solution, valid in the neighborhood of $z = 0$, as $w = Aw_1(z) + Bw_2(z)$. Note that

$$w_2(z) = w_1(z) \log z + 1 + z - \tfrac{1}{2}z^2 - \frac{7}{54}z^3 + \frac{37}{864}z^4 + \cdots + b_n z^n + \cdots.$$

Let us complete the picture by exhibiting another device for obtaining $w_2(z)$. If we substitute

$$w = \sum_{n=0}^{\infty} d_n z^{n+c}$$

into the original differential equation, we find that

$$z\frac{d^2w}{dz^2} + \frac{dw}{dz} + (1+z)w = c^2 d_0 z^{c-1} + [(1+c)^2 d_1 + d_0]z^c$$
$$+ \sum_{n=2}^{\infty} [(n+c)^2 d_n + d_{n-1} + d_{n-2}]z^{n+c-1}.$$

Hence, for the function

$$w_c = \sum_{n=0}^{\infty} d_n z^{n+c},$$

where the d_n satisfy the relations

$$(1+c)^2 d_1 + d_0 = 0,$$

and, for $n \geqq 2$,

$$d_n = -\frac{d_{n-1} + d_{n-2}}{(n+c)^2},$$

we see that

$$z\frac{d^2w_c}{dz^2} + \frac{dw_c}{dz} + (1+z)w_c = c^2 d_0 z^{c-1}.$$

At once we see that $w_2 = \left(\frac{\partial w_c}{\partial c}\right)_{c=0}$ is a solution of our differential equation. Now

$$\frac{\partial w_c}{\partial c} = \sum_{n=0}^{\infty} d_n z^{n+c} \log z + \sum_{n=0}^{\infty} \left(\frac{\partial d_n}{\partial c}\right) z^{n+c},$$

where the d_n are defined as above and the $\left(\frac{\partial d_n}{\partial c}\right)$ must satisfy the relations

$$(1+c)^2 \frac{\partial d_1}{\partial c} + \frac{\partial d_0}{\partial c} + 2(1+c)d_1 = 0,$$

and, for $n \geqq 2$,

$$\frac{\partial d_n}{\partial c} = -\frac{\dfrac{\partial d_{n-1}}{\partial c} + \dfrac{\partial d_{n-2}}{\partial c}}{(n+c)^2} + \frac{2(d_{n-1} + d_{n-2})}{(n+c)^3}$$

$$= -\frac{\dfrac{\partial d_{n-1}}{\partial c} + \dfrac{\partial d_{n-2}}{\partial c}}{(n+c)^2} - \frac{2d_n}{n+c}.$$

Let us now put $c = 0$ and, as examination of the relations readily shows is permissible, $(d_0)_{c=0} = 1, \left(\dfrac{\partial d_0}{\partial c}\right)_{c=0} = 1$. Then we have

$$w_2 = \sum_{n=0}^{\infty} (d_n)_{c=0}\, z^n \log z + \sum_{n=0}^{\infty} \left(\frac{\partial d_n}{\partial c}\right)_{c=0} z^n,$$

where $(d_0)_{c=0} = 1, \left(\dfrac{\partial d_0}{\partial c}\right)_{c=0} = 1$, and the remaining coefficients are determined by $(d_1)_{c=0} = -1, \left(\dfrac{\partial d_1}{\partial c}\right)_{c=0} = 1$, and, for $n \geqq 2$,

$$(d_n)_{c=0} = -\frac{(d_{n-1})_{c=0} + (d_{n-2})_{c=0}}{n^2},$$

$$\left(\frac{\partial d_n}{\partial c}\right)_{c=0} = -\frac{\left(\dfrac{\partial d_{n-1}}{\partial c}\right)_{c=0} + \left(\dfrac{\partial d_{n-2}}{\partial c}\right)_{c=0}}{n^2} - \frac{2(d_n)_{c=0}}{n}.$$

Direct comparison of definitions, the ones directly above with equations (51) and (52), shows that we may choose $(d_n)_{c=0} = a_n$, $\left(\dfrac{\partial d_n}{\partial c}\right)_{c=0} = b_n$, and that our new expression for $w_2(z)$ is then identical with the former one.

It is worth noticing that neither of the above two methods for determining $w_2(z)$ is as formidable in practice as might be suggested by the lengthy exposition which seems necessary to explain fully the processes involved in applying them. The essential nature of the processes involved is not adequately appreciated until the student has acquired reasonable proficiency and good technique by means of actual practice in the solution of specific problems.

PROBLEMS

1. Solve

$$2z\frac{d^2w}{dz^2}+\frac{dw}{dz}-(z+z^2)w=0.$$

Ans. $w = A\sum_{n=0}^{\infty} a_n z^{n+(1/2)} + B\sum_{n=0}^{\infty} b_n z^n$,

where the a_n and b_n may be determined as follows:

$a_0 = 1$, $a_1 = 0$, $a_2 = \frac{1}{10}$, and, for $n \geqq 3$,

$$a_n = \frac{a_{n-2}+a_{n-3}}{n(2n+1)};$$

$b_0 = 1$, $b_1 = 0$, $b_2 = \frac{1}{6}$, and, for $n \geqq 3$,

$$b_n = \frac{b_{n-2}+b_{n-3}}{n(2n-1)}.$$

2. Solve

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-z-1)w=0.$$

3. Solve the equation

$$z\frac{d^2w}{dz^2}-\frac{dw}{dz}+(1-z)w=0$$

by the method of this section. Compute several terms from your recurrence relation in each case, and obtain the results

$$w_1(z) = z^2 - \tfrac{1}{3}z^3 + \tfrac{1}{6}z^4 - \tfrac{1}{30}z^5 + \tfrac{1}{120}z^6 - \tfrac{1}{840}z^7 + \tfrac{1}{5040}z^8 - \tfrac{1}{45360}z^9 + \cdots,$$

$$w_2(z) = 1 + z + \tfrac{1}{3}z^3 - \tfrac{1}{24}z^4 + \tfrac{1}{40}z^5 - \tfrac{1}{360}z^6 + \tfrac{1}{1260}z^7 + \cdots.$$

Note that $w_1(z)$ may be written in more orderly fashion, indeed, that

$$w_1(z) = \frac{z^2}{1!} - \frac{z^3}{1!3} + \frac{z^4}{3!} - \frac{z^5}{3!5} + \frac{z^6}{5!} - \frac{z^7}{5!7} + \frac{z^8}{7!} - \frac{z^9}{7!9} + \cdots.$$

Thus guess a solution to your recurrence relation and then prove its validity by mathematical induction. That is, prove that

$$w_1(z) = \sum_{n=0}^{\infty} a_n z^{n+2},$$

where $a_0 = 1$, $a_1 = -\frac{1}{3}$, and, for $n \geqq 2$,

$$a_n = \frac{1}{(n+1)!} \quad \text{for } n \text{ even},$$

$$= -\frac{1}{n!(n+2)} \text{ for } n \text{ odd},$$

is a solution of the differential equation of this problem. Use this result to show that

$$w_1(z) = z \sinh z - \int_0^z \beta \sinh \beta \, d\beta$$

$$= z(\sinh z - \cosh z) + \sinh z.$$

Can you do anything of this sort with $w_2(z)$?

If you have not already done so, note that $w = e^z$ is a solution of the differential equation. Use this to complete the solution.

Ans. $w = Ae^z + Be^{-z}(2z + 1)$.

4. In using the exponent $c_2 = 0$ in Problem 3, you have found that a solution of the equation is given by

$$w = \sum_{n=0}^{\infty} a_n z^n$$

where a_0 and a_2 are arbitrary, $a_1 = a_0$, and, for $n \geqq 3$,

$$a_n = -\frac{a_{n-1} - a_{n-2}}{n(n-2)}.$$

Now choose $a_0 = 1$, $a_2 = \frac{1}{2}$ (instead of $a_2 = 0$ as before), and arrive at the corresponding solution.

5. Solve the following equation in the neighborhood of the ordinary point $z = 0$:

$$\frac{d^2w}{dz^2} + (1 + z + z^2)w = 0.$$

6. Solve the equation

$$\frac{d^2w}{dz^2} - 2\frac{dw}{dz} + 2w = 0$$

by elementary methods. Then use the methods of this section to obtain in particular a solution

$$w(z) = \sum_{n=0}^{\infty} a_n z^n,$$

in which $a_0 = 1$, $a_1 = 1$. Prove that you are thus obtaining, without multiplication of series, an expansion for $e^z \cos z$. See also page 77.

38. Remarks. In this chapter we have discussed solutions about regular singular points of second-order linear differential equations at some length. Sections 28 and 30 serve to suggest the kind of proof to be met in this subject. It seems unwise to carry through the details of further proofs of that kind in such a book as this. What is far more important to us in this first excursion into the less elementary portions of the subject of linear differential equations is the development of a reasonably good technique for actually carrying out the solution of numerical problems.

We repeat the admonition that multiplication of series is to be avoided when possible. To this end we always clear of fractions whenever we are using series methods on an equation with rational coefficients.

It has happened that our examples have exhibited the details of power-series solutions only where powers of the independent variables were being used. In other problems one point, which may appear obvious, should be watched. Suppose that we wish to obtain solutions of the equation

$$z^2(z-2)\frac{d^2w}{dz^2}+z\frac{dw}{dz}-3w=0$$

about the regular singular point $z = 2$. That is, we may wish to substitute

$$w=\sum_{n=0}^{\infty} a_n(z-2)^{n+c}$$

into the differential equation and determine c from the indicial equation and then a_n from the various recurrence relations. We should, before substituting the series into the equation, write the coefficients as series, even though finite, in $(z - 2)$. We may write $z^2 = (z-2)^2 + 4z - 4 = (z-2)^2 + 4(z-2) + 4$ and also $z = (z-2) + 2$. Hence the form of the equation which we use is

$$[(z-2)^3+4(z-2)^2+4(z-2)]\frac{d^2w}{dz^2}$$
$$+[(z-2)+2]\frac{dw}{dz}-3w=0.$$

In actual practice it seems advisable to let a change of independent variable do the above work for us. Thus, if we wish to solve

$$z^2(z-2)\frac{d^2w}{dz^2}+z\frac{dw}{dz}-3w=0$$

in the vicinity of $z = 2$, we put $z = 2 + t$ and solve the transformed equation in the vicinity of $t = 0$. That is, we solve

$$t(t^2 + 4t + 4)\frac{d^2w}{dt^2} + (t + 2)\frac{dw}{dt} - 3w = 0$$

in series of powers of t.

We shall make no explicit study of methods for obtaining particular integrals of non-homogeneous linear equations. Such solutions in power series are obtained in essentially the same way as power-series solutions of homogeneous linear equations. Indeed, we have already solved two problems of this type in obtaining the series for $w_3(z)$ in the numerical examples of Sections 35 and 37. Particular integrals are required from the student in Problems 9 and 10 of Section 78.

Finally we summarize a procedure which may be followed on meeting a new equation. First we locate and classify all singularities of the equation. Then, unless external circumstances dictate that we should confine our attention to specific regions, we obtain the indicial equation at each regular singular point. Next we determine to what case, Sections 31 to 34, our problem belongs at each regular singular point and proceed to obtain the desired solutions about each such point. If it is found necessary to discuss solutions in the neighborhood of an irregular singular point, the student will need to consult more advanced books, such as Forsyth's *Theory of Differential Equations*, to which we have already referred.

MISCELLANEOUS PROBLEMS

1. Obtain solutions about the ordinary point $z = 0$.

$$(1 - z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + 6w = 0.$$

2. Solve the equation of Problem 1 in the neighborhood of $z = 1$.

3. Solve the equation of Problem 1 in the neighborhood of $z = -1$.

4. Solve completely by series and from the series identify the solution in terms of elementary functions:

$$z^2\frac{d^2w}{dz^2} + z(1 + 2z)\frac{dw}{dz} + (z - \tfrac{1}{4})w = 0.$$

5. Solve Bessel's equation of index 1.

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - 1)w = 0.$$

6. Obtain solutions about $z = 0$.

$$2z(1 - z)\frac{d^2w}{dz^2} + (1 - z)\frac{dw}{dz} + 3w = 0.$$

7. Solve the equation of Problem 6 in the vicinity of $z = 1$.

8. Solve

$$z\frac{d^2w}{dz^2} + (1 + z)\frac{dw}{dz} + 2w = 0.$$

9. Obtain solutions about $z = 1$ for the equation

$$z(1 - z)\frac{d^2w}{dz^2} + (1 - 2z)\frac{dw}{dz} - \tfrac{1}{4}w = 0.$$

10. Solve the equation of Problem 9 in the vicinity of $z = 0$.

11. Obtain solutions in the neighborhood of $z = \infty$ ($\zeta = 0$) for the equation of Problem 1.

12. Obtain solutions about $z = 0$.

$$(1 - z)z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (2z - 1)w = 0.$$

13. Solve the equation of Problem 12 in the vicinity of $z = 1$.

14. Solve the equation

$$\frac{d^2w}{dz^2} + (1 + z + z^2 + z^3 + z^4)w = 0$$

in the vicinity of the ordinary point $z = 0$, and then compute several terms in each of $w_1(z)$ and $w_2(z)$.

15. Solve the equation

$$\frac{d^2w}{dz^2} + (z + e^z)w = 0$$

in the vicinity of the ordinary point $z = 0$, and then compute several terms in each of $w_1(z)$ and $w_2(z)$.

Hint: $w_1(z) = \sum_{n=0}^{\infty} a_n z^{n+1}$, where $a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{6}$, and, for $n \geqq 3$,

$$a_n = -\frac{a_{n-3}}{n(n + 1)} - \frac{1}{n(n + 1)}\sum_{k=0}^{n-2}\frac{a_{n-k-2}}{k!}.$$

16. Solve the equation

$$z^2\frac{d^2w}{dz^2} - (6 + z^3)w = 0.$$

CHAPTER V

EQUATIONS OF FUCHSIAN TYPE

39. Equation with all Singular Points Regular. An interesting equation, and one which we are now equipped to handle, is the most general second-order linear equation all of whose singular points are regular singular points. An equation of this kind is called an *equation of Fuchsian* type.* Note, however, than an equation of Fuchsian type is not necessarily of the second order; it may be of any order.

Consider the equation

$$\frac{d^2w}{dz^2} + P(z)\frac{dw}{dz} + Q(z)w = 0,$$

each of whose singular points is assumed to be a regular singular point. First we shall demonstrate that the number of singular points of this equation is finite. We know that at each singular point of the equation at least one of the two functions $P(z)$ and $Q(z)$ has a pole. If $P(z)$ has a pole at the point, it must be a pole of order one. If $Q(z)$ has a pole at the point, the pole may be of order one or two. Now, if the equation is to have an infinite number of regular singular points, at least one of $P(z)$ and $Q(z)$ must have an infinite number of poles. That infinity of poles automatically has a point of condensation, which may be at $z = \infty$, as it is for csc z. We now borrow from the theory of functions the fact that a condensation point of poles of a function is an essential singularity of that function. Of course, if either $P(z)$ or $Q(z)$ has an essential singularity at a point, that point is an irregular singular point of the differential equation. Thus we have proved that, if all the singular points of a linear differential equation of the second order are regular singular points, there must be only a finite number of such points.

We may now assume our equation to have exactly $(m + 1)$

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* Lazarus Fuchs (1833–1902) was a German mathematician who added a good deal to our knowledge of linear differential equations.

singular points, each a regular singular point. The discussion of Fuchsian equations with exactly one or with exactly two singular points is being left for the problems at the end of this section. Therefore we proceed with the assumption that $m > 1$. Now let the $(m + 1)$ singular points be $z = z_k$; $k = 1, 2, 3, \cdots, m$, and $z = \infty$. Then $P(z)$ can have no singularities in the finite plane except poles of order one at the points z_k iterated above. Hence

$$P(z) = \frac{P_1(z)}{(z - z_1)(z - z_2) \cdots (z - z_m)},$$

where $P_1(z)$ is a polynomial in z. We shall determine the maximum degree of $P_1(z)$ by using the fact that the equation has a regular singular point at $z = \infty$. Noting that $Q(z)$ can have no singularities in the finite plane except for poles of order one or two at z_k, we see that

$$Q(z) = \frac{Q_1(z)}{(z - z_1)^2(z - z_2)^2 \cdots (z - z_m)^2},$$

where $Q_1(z)$ is a polynomial in z. We shall determine the maximum degree of $Q_1(z)$ also by using the existence of a regular singular point at $z = \infty$.

In order to study the equation in the neighborhood of $z = \infty$, we put, as usual, $z = \dfrac{1}{\zeta}$ and study the transformed equation about $\zeta = 0$. Using the result in equation (39), page 79, we have at once

$$\frac{d^2w}{d\zeta^2} + \frac{1}{\zeta}\left[2 - \frac{P_1\left(\frac{1}{\zeta}\right)}{\zeta\left(\frac{1}{\zeta} - z_1\right)\left(\frac{1}{\zeta} - z_2\right)\cdots\left(\frac{1}{\zeta} - z_m\right)}\right]\frac{dw}{d\zeta}$$

$$+ \frac{1}{\zeta^2}\frac{Q_1\left(\frac{1}{\zeta}\right)}{\zeta^2\left(\frac{1}{\zeta} - z_1\right)^2\left(\frac{1}{\zeta} - z_2\right)^2\cdots\left(\frac{1}{\zeta} - z_m\right)^2} w = 0.$$

Therefore, in order to insure the existence of a regular singular point at $z = \infty$ or $\zeta = 0$, each of the functions

$$2 - \frac{P_1\left(\frac{1}{\zeta}\right)}{\zeta\left(\frac{1}{\zeta} - z_1\right)\left(\frac{1}{\zeta} - z_2\right)\cdots\left(\frac{1}{\zeta} - z_m\right)}$$

$$= 2 - \frac{\zeta^{m-1}P_1\left(\frac{1}{\zeta}\right)}{(1 - \zeta z_1)(1 - \zeta z_2)\cdots(1 - \zeta z_m)},$$

and

$$\frac{Q_1\left(\frac{1}{\zeta}\right)}{\zeta^2\left(\frac{1}{\zeta} - z_1\right)^2\left(\frac{1}{\zeta} - z_2\right)^2\cdots\left(\frac{1}{\zeta} - z_m\right)^2}$$

$$= \frac{\zeta^{2m-2}Q_1\left(\frac{1}{\zeta}\right)}{(1 - \zeta z_1)^2(1 - \zeta z_2)^2\cdots(1 - \zeta z_m)^2},$$

must be holomorphic at $\zeta = 0$. Evidently $P_1\left(\frac{1}{\zeta}\right)$ must be of degree $\leqq (m - 1)$ in $\left(\frac{1}{\zeta}\right)$ and $Q_1\left(\frac{1}{\zeta}\right)$ must be of degree $\leqq (2m - 2)$ in $\left(\frac{1}{\zeta}\right)$.

We have proved that $P_1(z)$ is a polynomial of degree $\leqq (m - 1)$ in z and that $Q_1(z)$ is a polynomial of degree $\leqq (2m - 2)$ in z. Thus we have proved the following theorem,* in which we use a more descriptive notation for the respective numerators.

Theorem 9. If a second-order linear differential equation has for its only singularities the $(m + 1)$ *regular singular points* $z = z_k$; $k = 1, 2, 3, \cdots, m$, *and* $z = \infty$, *then the equation may be written*

$$\frac{d^2w}{dz^2} + \frac{T_{(m-1)}(z)}{\psi(z)}\frac{dw}{dz} + \frac{T_{2(m-1)}(z)}{\psi^2(z)}w = 0, \tag{53}$$

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* For the general theorem due to Fuchs, see Forsyth's *Theory of Differential Equations*, Vol. IV, 1902, 125.

where

$$\psi(z) = (z - z_1)(z - z_2) \cdots (z - z_m)$$

and, for any subscript g, the symbol $T_g(z)$ denotes a polynomial of degree $\leqq g$ in z.

It is interesting to notice that the general theorem corresponding to Theorem 9 (general in the sense that the order of the equation is any integer n instead of the integer 2) leads to the equation

$$\frac{d^n w}{dz^n} + \sum_{i=1}^{n} \frac{T_{i(m-1)}(z)}{\psi^i(z)} \frac{d^{n-i} w}{dz^{n-i}} = 0,$$

in which we use the customary notation $\dfrac{d^0 w}{dz^0} = w$. This general theorem was proved by Fuchs in a memoir in *Crelle's Journal,*[*] vol. 68, 1868.

We know that,[†] since the degree of the numerator of $P(z)$ is less than the degree of the denominator of $P(z)$ it is possible to find constants A_k; $k = 1, 2, 3 \cdots, m$, such that

$$P(z) = \sum_{k=1}^{m} \frac{A_k}{z - z_k}.$$

Further, since the degree of the numerator of $Q(z)$ is less than the degree of the denominator of $Q(z)$, it is possible to find constants B_k and C_k; $k = 1, 2, 3, \cdots, m$, such that

$$Q(z) = \sum_{k=1}^{m} \frac{B_k}{(z - z_k)^2} + \sum_{k=1}^{m} \frac{C_k}{z - z_k}.$$

It is important to notice that the added restriction on $Q_1(z)$, the numerator of $Q(z)$, namely, that its degree is $\leqq (2m - 2)$

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[*] The official name of this periodical is *Journal für reine und angewandte Mathematik.* It is usually referred to as *Crelle's Journal,* however, in honor of its founder and first editor. August Leopold Crelle (1790–1855) founded the journal in question in 1826. Crelle was a minor German mathematician who often aided greatly young mathematicians whose genius transcended his own. Crelle's friendship and services for the brilliant young Norwegian Abel (see page 48) seem to have stood out among the few bright spots in Abel's life.

[†] From the elementary theory of partial fractions. See nearly any elementary textbook in calculus or college algebra.

rather than $\leqq (2m-1)$, is reflected in the fact that

$$\sum_{k=1}^{m} C_k = 0. \tag{54}$$

This is, of course, a direct result of the fact that, if the new form for $Q(z)$ be written as one fraction with its denominator identical with that in the old form of $Q(z)$, then the coefficient of z^{2m-1} in the numerator is precisely the indicated sum of the C_k's. We may now write equation (53) in what is for many purposes a much more convenient form,

$$\frac{d^2w}{dz^2} + \sum_{k=1}^{m} \frac{A_k}{z-z_k}\frac{dw}{dz} + \left[\sum_{k=1}^{m} \frac{B_k}{(z-z_k)^2} + \sum_{k=1}^{m} \frac{C_k}{z-z_k}\right] w = 0, \tag{55}$$

in which the A_k, B_k, and C_k; $k = 1, 2, 3, \cdots, m$, are arbitrary constants except for the restriction (54).

In the variable ζ we have

$$\frac{d^2w}{d\zeta^2} + \frac{1}{\zeta}\left[2 - \sum_{k=1}^{m} \frac{A_k}{1-\zeta z_k}\right]\frac{dw}{d\zeta} + \frac{1}{\zeta^2}\left[\sum_{k=1}^{m} \frac{B_k}{(1-\zeta z_k)^2} + \sum_{k=1}^{m} \frac{C_k}{\zeta(1-\zeta z_k)}\right] w = 0.$$

In view of the fact that

$$\frac{C_k}{\zeta(1-\zeta z_k)} = \frac{C_k}{\zeta} + \frac{C_k z_k}{1-\zeta z_k}$$

and

$$\sum_{k=1}^{m} \frac{C_k}{\zeta} = \frac{1}{\zeta}\sum_{k=1}^{m} C_k = 0,$$

we may use the form

$$\frac{d^2w}{d\zeta^2} + \frac{1}{\zeta}\left[2 - \sum_{k=1}^{m} \frac{A_k}{1-\zeta z_k}\right]\frac{dw}{d\zeta} + \frac{1}{\zeta^2}\left[\sum_{k=1}^{m} \frac{B_k}{(1-\zeta z_k)^2} + \sum_{k=1}^{m} \frac{C_k z_k}{1-\zeta z_k}\right] w = 0, \tag{56}$$

in which the restriction (54) still holds.

PROBLEMS

1. Obtain and solve the general linear equation of the second order with only two singularities, each of them a regular singular point.

2. Prove that: If a linear equation of the second order has only one singularity, that singularity must be an irregular singular point.

40. The Fuchsian Invariant. In equation (55) let the exponents (defined on page 87) at the points z_k be denoted by α_{1k} and α_{2k}. Further, let the exponents at $z = \infty$ be denoted by $\alpha_{1\infty}$ and $\alpha_{2\infty}$. Then, since the indicial equation corresponding to the regular singular point z_k is

$$c^2 + (A_k - 1)c + B_k = 0,$$

it is evident that, for $k = 1, 2, 3, \cdots, m$,

$$\begin{aligned} \alpha_{1k} + \alpha_{2k} &= 1 - A_k, \\ \alpha_{1k}\alpha_{2k} &= B_k. \end{aligned}$$

In the same way it follows from equation (56) that

$$\begin{aligned} \alpha_{1\infty} + \alpha_{2\infty} &= 1 - \left[2 - \sum_{k=1}^{m} A_k\right] = -1 + \sum_{k=1}^{m} A_k, \\ \alpha_{1\infty}\alpha_{2\infty} &= \sum_{k=1}^{m} (B_k + C_k z_k). \end{aligned}$$

We see that, if we obtain the sum of the exponents corresponding to all the singular points of our differential equation, we have

$$\begin{aligned} \alpha_{1\infty} + \alpha_{2\infty} + \sum_{k=1}^{m} (\alpha_{1k} + \alpha_{2k}) &= -1 + \sum_{k=1}^{m} A_k + \sum_{k=1}^{m} (1 - A_k) \\ &= m - 1, \end{aligned}$$

a result which depends only upon the number of singularities of equation (55). This is, for the equation of order two, the *Fuchsian invariant.*

For an equation* of order n, the corresponding Fuchsian invariant, the sum of the exponents at the regular singular points of an equation of Fuchsian type, is $\frac{1}{2}(m - 1)n(n - 1)$.

41. The Case of Three Regular Singular Points. From the equation

$$\text{(55)} \qquad \frac{d^2w}{dz^2} + \sum_{k=1}^{m} \frac{A_k}{z - z_k}\frac{dw}{dz} + \left[\sum_{k=1}^{m} \frac{B_k}{(z - z_k)^2} + \sum_{k=1}^{m} \frac{C_k}{z - z_k}\right] w = 0,$$

with the restriction

$$\text{(54)} \qquad \sum_{k=1}^{m} C_k = 0,$$

---

* See Fuchs and Forsyth, *loc. cit.*

we see that the second-order equation with $(m+1)$ regular singular points for its only singularities is completely determined except for $(3m-1)$ arbitrary constants. It is assumed that the singularities z_k; $k = 1, 2, 3, \cdots, m$, and $z = \infty$, have been chosen in advance. Now, if we choose the $(2m+2)$ exponents at the singular points with the restriction that the sum of these exponents is $(m-1)$, we may determine $(2m+1)$ of the arbitrary constants by means of the relations of the preceding section.

Since $3m - 1 = 2m + 1$, if $m = 2$, we see that, for a second-order linear equation with three regular singular points, z_1, z_2, and ∞, the equation is completely determined by the corresponding exponents α_{11}, α_{21}, α_{12}, α_{22}, $\alpha_{1\infty}$, and $\alpha_{2\infty}$, among which there is the necessary relation

$$\alpha_{11} + \alpha_{21} + \alpha_{12} + \alpha_{22} + \alpha_{1\infty} + \alpha_{2\infty} = 1.$$

This case is of particular interest. It leads to the famous hypergeometric equation of Gauss.*

For $m = 2$ we have, in addition to the Fuchsian invariant, the relations

$$\begin{aligned} A_1 &= 1 - \alpha_{11} - \alpha_{21}, \\ A_2 &= 1 - \alpha_{12} - \alpha_{22}, \\ B_1 &= \alpha_{11}\alpha_{21}, \\ B_2 &= \alpha_{12}\alpha_{22}, \\ B_1 + B_2 + C_1 z_1 + C_2 z_2 &= \alpha_{1\infty}\alpha_{2\infty}, \\ C_1 + C_2 &= 0. \end{aligned}$$

We may now write equation (55) explicitly in terms of the exponents whenever $m = 2$. The result is

$$\frac{d^2w}{dz^2} + \left[\frac{1 - \alpha_{11} - \alpha_{21}}{z - z_1} + \frac{1 - \alpha_{12} - \alpha_{22}}{z - z_2}\right]\frac{dw}{dz} \tag{57}$$

$$+ \left[\frac{\alpha_{11}\alpha_{21}}{(z - z_1)^2} + \frac{\alpha_{12}\alpha_{22}}{(z - z_2)^2} + \frac{\alpha_{1\infty}\alpha_{2\infty} - \alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22}}{(z - z_1)(z - z_2)}\right] w = 0.$$

---

* Karl Friedrich Gauss (1777–1855), a German mathematician, definitely one of the greatest in the history of the subject. His work touched upon practically every contemporary branch of pure and applied mathematics. His researches have had tremendous effect upon the subsequent development of the subjects.

PROBLEM

Discuss the Fuchsian equation of second order with singular points at z_1, z_2, z_3, and ∞. Put the equation into a form corresponding to (57) above, retaining the arbitrary parameter C_3.

Ans.
$$\frac{d^2w}{dz^2}+\left[\frac{1-\alpha_{11}-\alpha_{21}}{z-z_1}+\frac{1-\alpha_{12}-\alpha_{22}}{z-z_2}+\frac{1-\alpha_{13}-\alpha_{23}}{z-z_3}\right]\frac{dw}{dz}$$
$$+\left[\frac{\alpha_{11}\alpha_{21}}{(z-z_1)^2}+\frac{\alpha_{12}\alpha_{22}}{(z-z_2)^2}+\frac{\alpha_{13}\alpha_{23}}{(z-z_3)^2}+\frac{\alpha_{1\infty}\alpha_{2\infty}-\alpha_{11}\alpha_{21}-\alpha_{12}\alpha_{22}-\alpha_{13}\alpha_{23}}{(z-z_1)(z-z_2)}\right.$$
$$\left.+C_3\frac{(z_1-z_3)(z_2-z_3)}{(z-z_1)(z-z_2)(z-z_3)}\right]w=0.$$

42. The Riemann-Papperitz Equation. In order to introduce a notation standard in the literature on this subject, we shall obtain the equation of the second order with three regular singularities at any three points $z = a, b, c$ in the finite plane with exponents a' and a'' at $z = a$, b' and b'' at $z = b$, c' and c'' at $z = c$.

It appears to be simplest to obtain this equation from fundamental principles as we obtained equations (53) and (55). Following the line of argument of Section 39 we find that the equation assumes the form

$$\frac{d^2w}{dz^2}+\frac{P_2(z)}{(z-a)(z-b)(z-c)}\frac{dw}{dz}+\frac{Q_2(z)}{(z-a)^2(z-b)^2(z-c)^2}w=0,$$

where $P_2(z)$ and $Q_2(z)$ are polynomials. But this equation is to have $z = \infty$ as an ordinary point. Using $z = \dfrac{1}{\zeta}$, we may write

$$\frac{d^2w}{d\zeta^2}+\frac{1}{\zeta}\left[2-\frac{P_2\left(\frac{1}{\zeta}\right)}{\zeta\left(\frac{1}{\zeta}-a\right)\left(\frac{1}{\zeta}-b\right)\left(\frac{1}{\zeta}-c\right)}\right]\frac{dw}{d\zeta}$$
$$+\frac{1}{\zeta^2}\frac{Q_2\left(\frac{1}{\zeta}\right)}{\zeta^2\left(\frac{1}{\zeta}-a\right)^2\left(\frac{1}{\zeta}-b\right)^2\left(\frac{1}{\zeta}-c\right)^2}w=0.$$

This time the functions

$$\frac{1}{\zeta}\left[2-\frac{P_2\left(\frac{1}{\zeta}\right)}{\zeta\left(\frac{1}{\zeta}-a\right)\left(\frac{1}{\zeta}-b\right)\left(\frac{1}{\zeta}-c\right)}\right]$$

$$=\frac{2}{\zeta}-\frac{\zeta P_2\left(\frac{1}{\zeta}\right)}{(1-\zeta a)(1-\zeta b)(1-\zeta c)},$$

and

$$\frac{Q_2\left(\frac{1}{\zeta}\right)}{\zeta^4\left(\frac{1}{\zeta}-a\right)^2\left(\frac{1}{\zeta}-b\right)^2\left(\frac{1}{\zeta}-c\right)^2}=\frac{\zeta^2 Q_2\left(\frac{1}{\zeta}\right)}{(1-\zeta a)^2(1-\zeta b)^2(1-\zeta c)^2},$$

must be holomorphic at $\zeta = 0$.

It follows at once that $P_2\left(\frac{1}{\zeta}\right)$ must be a polynomial of degree two in $\left(\frac{1}{\zeta}\right)$ and that the coefficient of $\frac{1}{\zeta}$ in $P_2\left(\frac{1}{\zeta}\right)$ is 2. Further we see that $Q_2\left(\frac{1}{\zeta}\right)$ must be a polynomial of degree $\leqq 2$ in $\left(\frac{1}{\zeta}\right)$. In view of the degrees of $P_2(z)$ and $Q_2(z)$ there must exist constants A_1, A_2, A_3, B_1, B_2, and B_3, such that

$$\frac{P_2(z)}{(z-a)(z-b)(z-c)}=\frac{A_1}{z-a}+\frac{A_2}{z-b}+\frac{A_3}{z-c},$$

$$\frac{Q_2(z)}{(z-a)(z-b)(z-c)}=\frac{B_1}{z-a}+\frac{B_2}{z-b}+\frac{B_3}{z-c},$$

with, indeed, the added condition that $A_1 + A_2 + A_3 = 2$. This added condition is essentially due to the existence of the Fuchsian invariant. Note that this time we expand only a factor of the coefficient of w into partial fractions. This is possible because of the relatively low degree of $Q_2(z)$.

Our equation may be written

$$\frac{d^2w}{dz^2}+\left[\frac{A_1}{z-a}+\frac{A_2}{z-b}+\frac{A_3}{z-c}\right]\frac{dw}{dz}$$
$$+\left[\frac{B_1}{z-a}+\frac{B_2}{z-b}+\frac{B_3}{z-c}\right]\frac{w}{(z-a)(z-b)(z-c)}=0.$$

The indicial equations at $z = a$, $z = b$, $z = c$, are respectively

$$\rho^2 + (A_1 - 1)\rho + \frac{B_1}{(a-b)(a-c)} = 0,$$

$$\rho^2 + (A_2 - 1)\rho + \frac{B_2}{(b-a)(b-c)} = 0,$$

$$\rho^2 + (A_3 - 1)\rho + \frac{B_3}{(c-a)(c-b)} = 0,$$

where to avoid confusion we use ρ for the unknown in the indicial equation. Therefore, in terms of the exponents at the singularities we have

$$a' + a'' = 1 - A_1, \quad b' + b'' = 1 - A_2, \quad c' + c'' = 1 - A_3,$$

$$a'a'' = \frac{B_1}{(a-b)(a-c)}, \quad b'b'' = \frac{B_2}{(b-a)(b-c)},$$

$$c'c'' = \frac{B_3}{(c-a)(c-b)}.$$

Since $A_1 + A_2 + A_3 = 2$, we have

$$a' + a'' + b' + b'' + c' + c'' = 1,$$

which was to be expected.

We are finally in a position to write our equation in the form

$$\begin{aligned}(58)\quad \frac{d^2w}{dz^2} &+ \left[\frac{1 - a' - a''}{z-a} + \frac{1 - b' - b''}{z-b} + \frac{1 - c' - c''}{z-c}\right]\frac{dw}{dz} \\ &+ \left[\frac{a'a''(a-b)(a-c)}{z-a} + \frac{b'b''(b-a)(b-c)}{z-b} \right. \\ &\left. + \frac{c'c''(c-a)(c-b)}{z-c}\right]\frac{w}{(z-a)(z-b)(z-c)} = 0.\end{aligned}$$

This is the *Riemann-Papperitz equation.*[*]

PROBLEM

Obtain the Fuchsian equation of second order with singular points at $z = 0$, 1, -1, and with exponents α and β at each of those points.

Ans. $$\frac{d^2w}{dz^2} + \frac{2}{3}\frac{3z^2 - 1}{z(z^2-1)}\frac{dw}{dz} + \alpha(\tfrac{1}{3} - \alpha)\frac{3z^2+1}{z^2(z^2-1)^2}w = 0.$$

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[*] See E. Papperitz, *Mathematische Annalen*, **25**, 1885, 213, and Whittaker and Watson, 4th ed., 206.

43. The Riemann *P*-function. At times what is known as the Riemann *P*-symbol or *P*-function is used to signify that the function $w(z)$ satisfies equation (58). We write

$$w = P\begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix}$$

and mean by it neither more nor less than that w satisfies equation (58). In this symbol we write the singularities in the first row and place the exponents at each singularity directly below it. The fourth column is devoted to the independent variable. The above symbol is sometimes used with ∞ written in the place of c. Then what is meant is that $w(z)$ satisfies the equation which results from letting $c \to \infty$ in equation (58).

Two properties of Riemann's *P*-function are of outstanding importance in its use. First, if the transformation

$$z = \frac{At + B}{Ct + D}$$

transforms* the points $z = a$, $z = b$, $z = c$, respectively, into the points $t = a_1$, $t = b_1$, $t = c_1$, then

$$P\begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix} = P\begin{pmatrix} a_1 & b_1 & c_1 & \\ a' & b' & c' & t \\ a'' & b'' & c'' & \end{pmatrix}.$$

The above result may be proved directly, and with considerable tedious detail, by actually subjecting the independent variable in the Riemann-Papperitz equation to the suggested transformation. We omit proof of the property† in question.

A second property of the Riemann symbol is

$$(59)\quad P\begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix} = \left(\frac{z-a}{z-b}\right)^k P\begin{pmatrix} a & b & c & \\ a' - k & b' + k & c' & z \\ a'' - k & b'' + k & c'' & \end{pmatrix}.$$

---

* It seems easiest to obtain that transformation in the form

$$z - a = \frac{(a-b)(a-c)(c_1-b_1)(t-a_1)}{(a-b)(a_1-c_1)(t-b_1) + (a-c)(b_1-a_1)(t-c_1)}.$$

† See Whittaker and Watson, 4th ed., 207, for an argument to make this result intuitively plausible.

This property may be described as follows. If $w(z)$ satisfies

$$(58)\quad \frac{d^2w}{dz^2} + \left[\frac{1 - a' - a''}{z - a} + \frac{1 - b' - b''}{z - b} + \frac{1 - c' - c''}{z - c}\right]\frac{dw}{dz}$$
$$+ \left[\frac{a'a''(a - b)(a - c)}{z - a} + \frac{b'b''(b - a)(b - c)}{z - b}\right.$$
$$\left. + \frac{c'c''(c - a)(c - b)}{z - c}\right]\frac{w}{(z - a)(z - b)(z - c)} = 0,$$

and if we put

$$w(z) = \left(\frac{z - a}{z - b}\right)^k w_1(z),$$

then $w_1(z)$ satisfies an equation of the same form as (58), but with the exponents replaced by those indicated in the right member of property (59). We shall sketch a proof of (59).

There is associated with the singular point $z = a$ the solution

$$w = \sum_{n=0}^{\infty} a_n (z - a)^{n+a'}; \qquad a_0 \neq 0,$$

which belongs to the exponent a'. The corresponding function $w_1(z)$ is given by

$$w_1(z) = (z - b)^k (z - a)^{-k} \sum_{n=0}^{\infty} a_n (z - a)^{n+a'}$$
$$= (z - b)^k \sum_{n=0}^{\infty} a_n (z - a)^{n+a'-k}.$$

But $(z - b)^k$ is holomorphic at $z = a$, and there exists a series expansion

$$(z - b)^k = (a - b)^k + \sum_{n=1}^{\infty} b_n (z - a)^n.$$

Hence

$$w_1(z) = \sum_{n=0}^{\infty} c_n (z - a)^{n+a'-k},$$

where $c_0 \neq 0$. Thus to the exponent a' in the symbol for $w(z)$ there corresponds the exponent $(a' - k)$ in the symbol for $w_1(z)$. The other changes of exponent may be obtained in a like manner.

PROBLEM

Recall that the transformation $y = fv$ applied to $y'' + Py' + Qy = 0$ results in

$$v'' + \left(2\frac{f'}{f} + P\right)v' + \left(\frac{f''}{f} + P\frac{f'}{f} + Q\right)v = 0.$$

Note further that $\dfrac{f'}{f} = (\log f)'$ and that $\dfrac{f''}{f} = \left(\dfrac{f'}{f}\right)' + \left(\dfrac{f'}{f}\right)^2$. Now apply

$$w = \left(\frac{z-a}{z-b}\right)^k w_1$$

to equation (58) above. Show that the indicial equation at $z = a$ is transformed from

$$\rho^2 - (a' + a'')\rho + a'a'' = 0$$

into

$$\rho^2 - (a' + a'' - 2k)\rho + k^2 - (a' + a'')k + a'a'' = 0.$$

Obtain also the indicial equations at $z = b$ and $z = c$ for the equation in w_1. Thus prove the second property, (59), of the Riemann symbol.

44. A Canonical Form: the Hypergeometric Equation. Now that we have these two properties of the Riemann symbol, we may use them to reduce equation (58) to a relatively simple-looking form. Such a procedure is called, in many branches of mathematics, " reduction to a canonical form." We may put

$$w(z) = \left(\frac{z-a}{z-b}\right)^{a'} w_1(z),$$

and find that $w_1(z)$ satisfies the equation represented by the symbol

$$P\begin{pmatrix} a & b & c & \\ 0 & b' + a' & c' & z \\ a'' - a' & b'' + a' & c'' & \end{pmatrix}.$$

It is possible to get one other zero as an exponent for our equation. With this in mind we put

$$w_1(z) = \left(\frac{z-b}{z-c}\right)^{b'+a'} w_2(z),$$

and find that $w_2(z)$ satisfies the equation represented by

$$P\begin{pmatrix} a & b & c & \\ 0 & 0 & c' + b' + a' & z \\ a'' - a' & b'' - b' & c'' + b' + a' & \end{pmatrix}.$$

Now, since we have essentially only four exponents to deal with and since the sum of the exponents is still unity, we introduce three new parameters by which to designate the exponents. These are $\alpha = c' + b' + a'$, $\beta = c'' + b' + a'$, and $\gamma = 1 - a'' + a'$. Thus for $w(z)$ we may write

$$P\left(\begin{matrix} a & b & c & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right),$$

where, of course, to determine the non-zero exponent at $z = b$, we used the Fuchsian invariant. Finally, we subject z to a linear fractional transformation* which carries the points $z = a, b, c$ respectively into the points 0, 1, and ∞. If we use z again to designate the independent variable in the equation, we have arrived at our canonical form,

$$P\left(\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right).$$

We may now refer to

$$(57)\quad \frac{d^2w}{dz^2} + \left[\frac{1-\alpha_{11}-\alpha_{21}}{z-z_1} + \frac{1-\alpha_{12}-\alpha_{22}}{z-z_2}\right]\frac{dw}{dz}$$
$$\left[\frac{\alpha_{11}\alpha_{21}}{(z-z_1)^2} + \frac{\alpha_{12}\alpha_{22}}{(z-z_2)^2} + \frac{\alpha_{1\infty}\alpha_{2\infty} - \alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22}}{(z-z_1)(z-z_2)}\right]w = 0$$

to write down our canonical form for the general linear equation of the second order with exactly three singular points each of them regular. The result is the (mathematically) celebrated *hypergeometric equation,* which has been the nucleus of a great many mathematical papers. This equation is

$$\frac{d^2w}{dz^2} + \left[\frac{1-(1-\gamma)}{z} + \frac{1-(\gamma-\alpha-\beta)}{z-1}\right]\frac{dw}{dz} + \frac{\alpha\beta}{z(z-1)}w = 0,$$

or

$$(60)\quad z(1-z)\frac{d^2w}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{dw}{dz} - \alpha\beta w = 0.$$

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* See the transformation on page 65.

CHAPTER VI

THE HYPERGEOMETRIC EQUATION

45. The Hypergeometric Series or Function. Consider now the hypergeometric equation

$$(60) \qquad z(1-z)\frac{d^2w}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{dw}{dz} - \alpha\beta w = 0.$$

We know already that at the regular singular point $z = 0$ the exponents of this equation are 0 and $(1 - \gamma)$. Let us use the former exponent and put

$$w = \sum_{n=0}^{\infty} a_n z^n.$$

Direct substitution into the equation yields the identity

$$\sum_{n=0}^{\infty} n(n-1)a_n z^{n-1} - \sum_{n=0}^{\infty} n(n-1)a_n z^n + \sum_{n=0}^{\infty} n\gamma a_n z^{n-1}$$
$$- \sum_{n=0}^{\infty} n(\alpha + \beta + 1)a_n z^n - \sum_{n=0}^{\infty} \alpha\beta a_n z^n = 0.$$

We group corresponding summations and find

$$\sum_{n=0}^{\infty} n(\gamma + n - 1)a_n z^{n-1} - \sum_{n=0}^{\infty} [n^2 + n(\alpha + \beta) + \alpha\beta]a_n z^n = 0.$$

Now this may be written

$$\sum_{n=0}^{\infty} n(\gamma + n - 1)a_n z^{n-1} - \sum_{n=0}^{\infty} (\alpha + n)(\beta + n)a_n z^n = 0,$$

or

$$\sum_{n=1}^{\infty} n(\gamma + n - 1)a_n z^{n-1}$$
$$- \sum_{n=1}^{\infty} (\alpha + n - 1)(\beta + n - 1)a_{n-1} z^{n-1} = 0.$$

Thus we find that the equations for the determination of the a_n are

$$a_n = \frac{(\alpha + n - 1)(\beta + n - 1)}{(\gamma + n - 1)\cdot n}\, a_{n-1},$$

for $n \geqq 1$, with a_0 arbitrary. At this stage we must insist that γ be neither zero nor a negative integer, so that none of the denominators above will turn out to be zero. With this assumption we may solve the above recurrence relation easily. We find

$$a_n = \frac{\alpha(\alpha + 1)\cdots(\alpha + n - 1)\beta(\beta + 1)\cdots(\beta + n - 1)}{\gamma(\gamma + 1)\cdots(\gamma + n - 1)n!}\, a_0,$$

and we may choose $a_0 = 1$ in order to get a particular solution.

In advanced calculus we found that the gamma function $\Gamma(x)$ has the property

$$\Gamma(x + 1) = x\Gamma(x).$$

Hence

$$\begin{aligned}\Gamma(\alpha + n) &= (\alpha + n - 1)\Gamma(\alpha + n - 1)\\ &= (\alpha + n - 1)(\alpha + n - 2)\Gamma(\alpha + n - 2)\\ &= \cdots = (\alpha + n - 1)(\alpha + n - 2)\cdots(\alpha + 1)(\alpha)\Gamma(\alpha).\end{aligned}$$

Thus we may write

$$a_n = \frac{\Gamma(\alpha + n)\Gamma(\beta + n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma + n)}\,\frac{1}{n!}.$$

We have found $w(z)$ in the form

$$w(z) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\alpha + n)\Gamma(\beta + n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma + n)}\,\frac{z^n}{n!}.$$

The determination of the region of convergence of this series will be left for the problems. This function occurs so frequently in mathematical analysis that a special symbol $F(\alpha, \beta, \gamma, z)$ is used for it. We call it the *hypergeometric series** or the *hypergeometric*

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* Euler seems to have been the first to study this series and to point out that it is a solution of equation (60). Many years later Gauss came along and obtained many other results relating to this series.

function, and we write it in any one of the three equivalent forms:

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n)\Gamma(\beta + n)}{\Gamma(\gamma + n)} \frac{z^n}{n!}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1)\beta(\beta + 1) \cdots (\beta + n - 1)}{\gamma(\gamma + 1) \cdots (\gamma + n - 1)} \frac{z^n}{n!}$$

$$= 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)}\frac{z^2}{2!} + \cdots +$$

$$\frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1)\beta(\beta + 1) \cdots (\beta + n - 1)}{\gamma(\gamma + 1) \cdots (\gamma + n - 1)} \frac{z^n}{n!} + \cdots .$$

Of course, any solution of equation (60) is a hypergeometric function, but we reserve the definitive expression the hypergeometric function for $F(\alpha, \beta, \gamma, z)$. As we proceed in this chapter our interest will oscillate, being on occasion attached to the hypergeometric function rather than to the hypergeometric equation.

46. A Second Solution about $z = 0$. Let us now return to

$$(60) \quad z(1 - z)\frac{d^2w}{dz^2} + [\gamma - (\alpha + \beta + 1) z]\frac{dw}{dz} - \alpha\beta w = 0,$$

and find the solution which we know exists in the form

$$w = \sum_{n=0}^{\infty} b_n z^{n+1-\gamma}.$$

Of course, we are assuming here that we have the unexceptional case; that is, that the difference of the exponents at $z = 0$ is neither zero nor a positive integer. But the difference of those exponents is either $(\gamma - 1)$ or $(1 - \gamma)$, and we have already excluded zero and negative integers from the set of permissible values of γ. It is clear that all we need to do now is to exclude the positive integers.

Proceeding toward the desired solution, we find that

$$\sum_{n=0}^{\infty} (n + 1 - \gamma)(n - \gamma)b_n z^{n-\gamma} - \sum_{n=0}^{\infty} (n + 1 - \gamma)(n - \gamma)b_n z^{n+1-\gamma}$$
$$+ \sum_{n=0}^{\infty} \gamma(n + 1 - \gamma)b_n z^{n-\gamma}$$
$$- \sum_{n=0}^{\infty} (\alpha + \beta + 1)(n + 1 - \gamma)b_n z^{n+1-\gamma} - \sum_{n=0}^{\infty} \alpha\beta b_n z^{n+1-\gamma} = 0.$$

Next we have

$$\sum_{n=0}^{\infty} n(n+1-\gamma)b_n z^{n-\gamma}$$
$$-\sum_{n=0}^{\infty} [(n+1-\gamma)^2 + (\alpha+\beta)(n+1-\gamma) + \alpha\beta]b_n z^{n+1-\gamma} = 0,$$

or

$$\sum_{n=1}^{\infty} n(n+1-\gamma)b_n z^{n-\gamma}$$
$$-\sum_{n=0}^{\infty} (n+1-\gamma+\alpha)(n+1-\gamma+\beta)b_n z^{n+1-\gamma} = 0.$$

Shifting the index in the latter summation we obtain

$$\sum_{n=1}^{\infty} n(n+1-\gamma)b_n z^{n-\gamma}$$
$$-\sum_{n=1}^{\infty} (n-\gamma+\alpha)(n-\gamma+\beta)b_{n-1} z^{n-\gamma} = 0.$$

Thus the b_n are to be determined by

$$b_n = \frac{(n-\gamma+\alpha)(n-\gamma+\beta)}{(n+1-\gamma)\cdot n} b_{n-1},$$

for $n \geqq 1$, with b_0 arbitrary.

The marked similarity of the above equations to the equations for the determination of the a_n (see page 145) prompts us to attempt to identify the two systems. This is easily done. Let us rewrite our equations in the form

$$b_n = \frac{(\alpha-\gamma+1+n-1)(\beta-\gamma+1+n-1)}{(2-\gamma+n-1)\cdot n} b_{n-1}.$$

Now we see that, if in the work of the preceding section we replace α by $(\alpha-\gamma+1)$, β by $(\beta-\gamma+1)$ and γ by $(2-\gamma)$, we shall again satisfy all our conditions and have a solution to equation (60). Thus our second solution is

$$w = \frac{\Gamma(2-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \times$$
$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha-\gamma+1+n)\Gamma(\beta-\gamma+1+n)}{\Gamma(2-\gamma+n)} \frac{z^{n+1-\gamma}}{n!},$$

or, in standard notation,

$$w = z^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma, z).$$

If we combine the restrictions of this and the preceding section we see that γ must not be an integer (positive, negative, or zero). With these restrictions imposed, we may write the general solution of the hypergeometric equation as

$$w = AF(\alpha, \beta, \gamma, z) + Bz^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2-\gamma, z),$$

where A and B are arbitrary constants.

47. The Exceptional Values of γ. The results in the previous two sections are, as we have seen, valid for all γ except integral values, positive, negative, or zero. If $\gamma = 1$, the indicial equation has equal roots and we are confronted with Case II, of Chapter IV (page 96). Each of the other exceptional values of γ must lead to either Case III or Case IV of Chapter IV. We leave it to the student to discuss these, with the caution that the separation into problems below is artificial except for the singling out of $\gamma = 1$.

PROBLEMS

1. Solve the hypergeometric equation,

$$(60) \qquad z(1 - z)\frac{d^2w}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{dw}{dz} - \alpha\beta w = 0$$

when $\gamma = 1$.

2. Solve the hypergeometric equation when $\gamma = 0$.

3. Solve the hypergeometric equation when $\gamma = -m$, m a positive integer.

4. Solve the hypergeometric equation when $\gamma = m$, m an integer > 1.

48. Certain Transformations of the Equation. Interesting results* are obtained by studying certain transformations by which the hypergeometric equation,

$$(60) \quad z(1 - z)\frac{d^2w_1}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{dw_1}{dz} - \alpha\beta w_1 = 0,$$

may be transformed into an equation of precisely the same form but with different parameters, namely,

$$(61) \quad t(1 - t)\frac{d^2w_2}{dt^2} + [\gamma' - (\alpha' + \beta' + 1)t]\frac{dw_2}{dt} - \alpha'\beta' w_2 = 0.$$

We intend first to determine α', β', and γ' in terms of α, β, and

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* The student who wishes to know where he is heading at this stage should consult Section 51 below.

γ, under the assumption that equation (60) may be transformed into equation (61) by means of the transformations

$$w_1 = v(z)w_2,$$
$$t = t(z),$$

in which $t(z)$ is to be independent of the parameters.

We shall use the results of Section 7, which it would be wise for the student to review now. The equivalence of equations (60) and (61) under such transformations is dependent, in our present notation, upon the satisfaction of the equations

$$\left(\frac{dt}{dz}\right)^2 I_2(t) = I_1(z) - \tfrac{1}{2}\{t, z\},$$
$$u_1 = \left(\frac{dz}{dt}\right)^{1/2} u_2,$$

where I_1, I_2, u_1, and u_2 are as indicated in the normal forms,

$$\frac{d^2u_1}{dz^2} + I_1(z)u_1 = 0,$$
$$\frac{d^2u_2}{dt^2} + I_2(t)u_2 = 0,$$

respectively, of equations (60) and (61).

Let us proceed to the determination of $v(z)$. We first note that

$$\exp\left[-\frac{1}{2}\int\frac{\gamma - (\alpha + \beta + 1)z}{z(1 - z)}\,dz\right] = z^{-\gamma/2}(1 - z)^{-(\alpha+\beta+1-\gamma)/2}$$

so that

$$w_1 = z^{-\gamma/2}(1 - z)^{-(\alpha+\beta+1-\gamma)/2}u_1.$$

Considerations of symmetry yield

$$w_2 = t^{-\gamma'/2}(1 - t)^{-(\alpha'+\beta'+1-\gamma')/2}u_2.$$

But we already have a relation between u_1 and u_2. Hence we see that

$$(62)\quad w_1 = z^{-\gamma/2}(1 - z)^{-(\alpha+\beta+1-\gamma)/2}t^{\gamma'/2}(1 - t)^{(\alpha'+\beta'+1-\gamma')/2}\left(\frac{dz}{dt}\right)^{1/2} w_2.$$

Equation (62) contains the explicit expression for the $v(z)$ of

$w_1 = v(z)w_2$ in terms of z and t. We now turn to the determination of $t(z)$.

It may be shown by direct computation that the invariant of

$$(60) \qquad z(1-z)\frac{d^2w_1}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{dw_1}{dz} - \alpha\beta w_1 = 0$$

is

$$I_1(z) = \frac{Az^2 + Bz + C}{4z^2(1-z)^2},$$

where

$$\begin{aligned} A &= 1 - (\alpha - \beta)^2, \\ B &= (1-\gamma)^2 + (\alpha-\beta)^2 - (\gamma-\alpha-\beta)^2 - 1, \\ C &= 1 - (1-\gamma)^2. \end{aligned}$$

For equation (61) it follows similarly that we have

$$I_2(t) = \frac{A't^2 + B't + C'}{4t^2(1-t)^2},$$

where

$$\begin{aligned} A' &= 1 - (\alpha' - \beta')^2, \\ B' &= (1-\gamma')^2 + (\alpha'-\beta')^2 - (\gamma'-\alpha'-\beta')^2 - 1, \\ C' &= 1 - (1-\gamma')^2. \end{aligned}$$

Since we insist that $t(z)$ be independent of the parameters and since $I_1(z)$ and $I_2(t)$ depend upon the parameters, we shall set $\{t, z\} = 0$ and then use the resulting equation,

$$(63) \qquad \left(\frac{dt}{dz}\right)^2 I_2(t) = I_1(z),$$

to determine α', β', and γ' in terms of α, β, and γ.

Let us solve $\{t, z\} = 0$. If we refer to the section on the Schwarzian Derivative, we find on page 18 the result

$$\left\{\frac{ax+b}{cx+d}, x\right\} = 0.$$

Hence

$$(64) \qquad t = \frac{az+b}{cz+d},$$

a linear fractional expression in z with a, b, c, and d arbitrary constants, is a solution of $\{t, z\} = 0$. There are essentially only three arbitrary constants in (64), of course.

PROBLEM

Consider the equation

$$\{t, z\} = \frac{t'''}{t'} - \frac{3}{2}\frac{(t'')^2}{(t')^2} = 0.$$

Rewrite it as

$$(t')^{-3/2}t''' - \tfrac{3}{2}(t')^{-5/2}(t'')^2 = 0,$$

an exact equation. Integrate this equation, and note that its first integral is also exact. Complete the solution of $\{t, z\} = 0$, and thus derive the solution (64) above.

49. Determination of the Linear Fractional Transformation. From (64) we have

$$\frac{dt}{dz} = \frac{ad - bc}{(cz + d)^2}.$$

Hence we may write equation (63) as

$$\frac{(ad - bc)^2}{(cz + d)^4}\,\frac{A'\left(\dfrac{az + b}{cz + d}\right)^2 + B'\left(\dfrac{az + b}{cz + d}\right) + C'}{4\left(\dfrac{az + b}{cz + d}\right)^2\left[1 - \dfrac{az + b}{cz + d}\right]^2} = \frac{Az^2 + Bz + C}{4z^2(1 - z)^2}.$$

This may be simplified somewhat and put in the form

$$\frac{(ad - bc)^2[A'(az + b)^2 + B'(az + b)(cz + d) + C'(cz + d)^2]}{(az + b)^2(cz + d)^2[(c - a)z + (d - b)]^2} = \frac{Az^2 + Bz + C}{z^2(1 - z)^2}.$$

Let us now assume that $C \neq 0$ and that $A + B + C \neq 0$. These restrictions are equivalent to $(1 - \gamma)^2 \neq 1$ and $(\gamma - \alpha - \beta)^2 \neq 1$. They permit us to say that the numerator of the right member of the above identity does not vanish at $z = 0$ and does not vanish at $z = 1$. Under these conditions that right member has a pole of order two at $z = 0$ and a pole of order two at $z = 1$, and has no other singularities. This must also be true of the left member of the identity. Hence the two denominators must be equal except for a constant non-zero factor, which we shall call m. We have arrived at the identity

$$(65)\quad (az + b)^2(cz + d)^2[(c - a)z + (d - b)]^2 = mz^2(1 - z)^2.$$

First let us consider the coefficient of z^6 in the left member of (65). That coefficient must vanish, and the constant term in that member must vanish. Thus we have

$$a^2c^2(c - a)^2 = 0,$$
$$b^2d^2(d - b)^2 = 0.$$

These two equations appear to give us nine cases for consideration. But three of these cases are absurd in view of the fact that $m \neq 0$. The three impossible cases are

$$\begin{aligned} a = b &= 0 \\ c = d &= 0, \\ c - a = d - b &= 0. \end{aligned}$$

We are left with the following six possibilities:

Case I: $a = 0, \quad d = 0.$
Case II: $a = 0, \quad d = b.$
Case III: $c = 0, \quad b = 0.$
Case IV: $c = 0, \quad d = b.$
Case V: $c = a, \quad b = 0.$
Case VI: $c = a, \quad d = 0.$

We now rewrite the identity (65) in the form to which it reduces in each case.

Case I: $b^2c^2z^2(cz - b)^2 = mz^2(1 - z)^2.$
Case II: $b^2c^2z^2(cz + b)^2 = mz^2(1 - z)^2.$
Case III: $a^2d^2z^2(az - d)^2 = mz^2(1 - z)^2.$
Case IV: $a^2b^2z^2(az + b)^2 = mz^2(1 - z)^2.$
Case V: $a^2d^2z^2(az + d)^2 = mz^2(1 - z)^2.$
Case VI: $a^2b^2z^2(az + b)^2 = mz^2(1 - z)^2.$

The identities are now so simple that the solutions follow at once by equating coefficients of like powers of z.

Case I:
$m = b^2c^4, \quad m = b^3c^3, \quad m = b^4c^2.$ Hence $b = c, \quad m = b^6.$

Case II:
$m = b^2c^4, \quad m = -b^3c^3, \quad m = b^4c^2.$ Hence $b = -c, \quad m = b^6.$
Case III:
$m = a^4d^2, \quad m = a^3d^3, \quad m = a^2d^4.$ Hence $a = d, \quad m = a^6.$
Case IV:
$m = a^4b^2, \quad m = -a^3b^3, \quad m = a^2b^4.$ Hence $a = -b, \quad m = a^6.$
Case V:
$m = a^4d^2, \quad m = -a^3d^3, \quad m = a^2d^4.$ Hence $a = -d, \quad m = a^6.$
Case VI:
$m = a^4b^2, \quad m = -a^3b^3, \quad m = a^2b^4.$ Hence $a = -b, \quad m = a^6.$

We are now able to write down t in terms of z in each of the six cases.

Case I: $a = 0, \quad c = b, \quad d = 0.$

$$t = \frac{b}{bz} = \frac{1}{z}.$$

Case II: $a = 0, \quad c = -b, \quad d = b.$

$$t = \frac{b}{-bz + b} = \frac{1}{1 - z}.$$

Case III: $b = 0, \quad c = 0, \quad d = a.$

$$t = \frac{az}{a} = z.$$

Case IV: $b = -a, \quad c = 0, \quad d = -a.$

$$t = \frac{az - a}{-a} = 1 - z.$$

Case V: $b = 0, \quad c = a, \quad d = -a.$

$$t = \frac{az}{az - a} = \frac{z}{z - 1}.$$

Case VI: $b = -a, \quad c = a, \quad d = 0.$

$$t = \frac{az - a}{az} = \frac{z - 1}{z}.$$

We have shown that t must be one of the six functions

$$\frac{1}{z}, \quad \frac{1}{1 - z}, \quad z, \quad 1 - z, \quad \frac{z}{z - 1}, \quad \frac{z - 1}{z},$$

which are the elements of the very well-known cross-ratio group. See page 46.

50. The Final Determination of α', β', and γ'. Since we now know a, b, c, d, and m in each case in terms of some one of themselves, we may return to the identity immediately preceding (65) and put the numerator of the left member equal to m times the numerator of the right member. This will permit us to determine A', B', and C' in terms of A, B, and C and hence, finally, α', β', and γ' in terms of α, β, and γ.

Case I: $a = 0$, $c = b$, $d = 0$, $m = b^6$. Then $(ad - bc)^2 = b^4$, and the identity

$$(ad - bc)^2[A'(az + b)^2 + B'(az + b)(cz + d) + C'(cz + d)^2]$$
$$= m(Az^2 + Bz + C)$$

becomes

$$A' + B'z + C'z^2 = Az^2 + Bz + C.$$

In precisely the same manner we find identities in each of the other five cases. They are:

Case II: $A' + B'(1 - z) + C'(1 - z)^2 = Az^2 + Bz + C$.
Case III: $A'z^2 + B'z + C' = Az^2 + Bz + C$.
Case IV: $A'(z - 1)^2 - B'(z - 1) + C' = Az^2 + Bz + C$.
Case V: $A'z^2 + B'z(z - 1) + C'(z - 1)^2 = Az^2 + Bz + C$.
Case VI: $A'(z - 1)^2 + B'z(z - 1) + C'z^2 = Az^2 + Bz + C$.

In each instance it is a simple matter to determine A', B', and C' in terms of A, B, and C. The results follow.

Case I: $A' = C$, $B' = B$, $C' = A$.
Case II: $A' = A + B + C$, $B' = -2A - B$, $C' = A$.
Case III: $A' = A$, $B' = B$, $C' = C$.
Case IV: $A' = A$, $B' = -2A - B$, $C' = A + B + C$.
Case V: $A' = A + B + C$, $B' = -B - 2C$, $C' = C$.
Case VI: $A' = C$, $B' = -B - 2C$, $C' = A + B + C$.

We are now in a position to write the equations for the determination of α', β', and γ' in terms of α, β, and γ in each case. We use the expressions on page 150 for A, B, and C in terms of α, β, and γ.

Case I: The three equations first appear as

$$1-(\alpha'-\beta')^2=1-(1-\gamma)^2,$$
$$(1-\gamma')^2+(\alpha'-\beta')^2-(\gamma'-\alpha'-\beta')^2-1=(1-\gamma)^2+(\alpha-\beta)^2-(\gamma-\alpha-\beta)^2-1,$$
$$1-(1-\gamma')^2=1-(\alpha-\beta)^2.$$

However, it is easy to see that this reduces to the equivalent system

$$(\alpha'-\beta')^2=(1-\gamma)^2,$$
$$(1-\gamma')^2=(\alpha-\beta)^2,$$
$$(\gamma'-\alpha'-\beta')^2=(\gamma-\alpha-\beta)^2.$$

By taking square roots in each case we can reduce this system of quadratic equations to eight systems each containing three linear equations. For example, two such systems are

$$\alpha'-\beta'=1-\gamma,$$
$$1-\gamma'=\alpha-\beta,$$
$$\gamma'-\alpha'-\beta'=\gamma-\alpha-\beta,$$

and

$$\alpha'-\beta'=-1+\gamma,$$
$$1-\gamma'=\alpha-\beta,$$
$$\gamma'-\alpha'-\beta'=\gamma-\alpha-\beta.$$

The details of the solution, then, are simple. We restrict ourselves to the statement of the results. The eight systems of equations have the following solutions:

I*a*: $\alpha'=\alpha,\quad \beta'=\alpha-\gamma+1,\quad \gamma'=\alpha-\beta+1.$
I*b*: $\alpha'=\beta,\quad \beta'=\beta-\gamma+1,\quad \gamma'=\beta-\alpha+1.$
I*c*: $\alpha'=1-\alpha,\quad \beta'=\gamma-\alpha,\quad \gamma'=\beta-\alpha+1.$
I*d*: $\alpha'=1-\beta,\quad \beta'=\gamma-\beta,\quad \gamma'=\alpha-\beta+1.$
I*e*: $\alpha'=\alpha-\gamma+1,\quad \beta'=\alpha,\quad \gamma'=\alpha-\beta+1.$
I*f*: $\alpha'=\beta-\gamma+1,\quad \beta'=\beta,\quad \gamma'=\beta-\alpha+1.$
I*g*: $\alpha'=\gamma-\alpha,\quad \beta'=1-\alpha,\quad \gamma'=\beta-\alpha+1.$
I*h*: $\alpha'=\gamma-\beta,\quad \beta'=1-\beta,\quad \gamma'=\alpha-\beta+1.$

It may be seen that in this statement of results those in cases I*b* and I*f* correspond to the sample equations given directly above.

Now it is high time that we noted the fact that the hypergeometric equation is invariant under an interchange of α and β, and in particular that $F(\alpha, \beta, \gamma, z) = F(\beta, \alpha, \gamma, z)$. With these facts in mind we see that cases I*e*, I*f*, I*g*, and I*h* are respectively equivalent to cases I*a*, I*b*, I*c*, and I*d*. Naturally this same phenomenon is to be found in the discussion of the other major separations. In each we proceed as was indicated in detail for Case I. Omitting redundant cases the results are shown in the following table:

CASE	α'	β'	γ'
II*a*	α	$\gamma - \beta$	$\alpha - \beta + 1$
II*b*	β	$\gamma - \alpha$	$\beta - \alpha + 1$
II*c*	$\alpha - \gamma + 1$	$1 - \beta$	$\alpha - \beta + 1$
II*d*	$\beta - \gamma + 1$	$1 - \alpha$	$\beta - \alpha + 1$
III*a*	α	β	γ
III*b*	$\gamma - \alpha$	$\gamma - \beta$	γ
III*c*	$\alpha - \gamma + 1$	$\beta - \gamma + 1$	$2 - \gamma$
III*d*	$1 - \alpha$	$1 - \beta$	$2 - \gamma$
IV*a*	α	β	$\alpha + \beta - \gamma + 1$
IV*b*	$\alpha - \gamma + 1$	$\beta - \gamma + 1$	$\alpha + \beta - \gamma + 1$
IV*c*	$\gamma - \alpha$	$\gamma - \beta$	$\gamma - \alpha - \beta + 1$
IV*d*	$1 - \alpha$	$1 - \beta$	$\gamma - \alpha - \beta + 1$
V*a*	α	$\gamma - \beta$	γ
V*b*	β	$\gamma - \alpha$	γ
V*c*	$\alpha - \gamma + 1$	$1 - \beta$	$2 - \gamma$
V*d*	$\beta - \gamma + 1$	$1 - \alpha$	$2 - \gamma$
VI*a*	α	$\alpha - \gamma + 1$	$\alpha + \beta - \gamma + 1$
VI*b*	β	$\beta - \gamma + 1$	$\alpha + \beta - \gamma + 1$
VI*c*	$1 - \alpha$	$\gamma - \alpha$	$\gamma - \alpha - \beta + 1$
VI*d*	$1 - \beta$	$\gamma - \beta$	$\gamma - \alpha - \beta + 1$

PROBLEM

Carry through the details in the determination of α', β', and γ' in some particular case, such as Case IV.

51. Kummer's Twenty-Four Solutions. Reviewing the developments of the preceding three sections, we find that we have attained the following goal. We can now write down twenty-four solutions of the hypergeometric equation, each of those solutions corresponding to one of our cases I*a* to VI*d*. In order to do this we write $w_2 = F(\alpha', \beta', \gamma', t)$ in

(62) $$w_1 = z^{-\gamma/2}(1-z)^{-(\alpha+\beta+1-\gamma)/2}t^{\gamma'/2}(1-t)^{(\alpha'+\beta'+1-\gamma')/2}\left(\frac{dz}{dt}\right)^{1/2} w_2,$$

and use for α', β', γ', and t the expressions which we have obtained for them in terms of α, β, γ, and z in each case. The result, $w_1(z)$, is a solution of the hypergeometric equation.* We now list those solutions, disregarding any constant factor which may enter:

I*a*: $z^{-\alpha}F\left(\alpha, \alpha-\gamma+1, \alpha-\beta+1, \frac{1}{z}\right).$

I*b*: $z^{-\beta}F\left(\beta, \beta-\gamma+1, \beta-\alpha+1, \frac{1}{z}\right).$

I*c*: $z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}F\left(1-\alpha, \gamma-\alpha, \beta-\alpha+1, \frac{1}{z}\right).$

I*d*: $z^{\beta-\gamma}(1-z)^{\gamma-\alpha-\beta}F\left(1-\beta, \gamma-\beta, \alpha-\beta+1, \frac{1}{z}\right).$

II*a*: $(1-z)^{-\alpha}F\left(\alpha, \gamma-\beta, \alpha-\beta+1, \frac{1}{1-z}\right).$

II*b*: $(1-z)^{-\beta}F\left(\beta, \gamma-\alpha, \beta-\alpha+1, \frac{1}{1-z}\right).$

II*c*: $z^{1-\gamma}(1-z)^{\gamma-\alpha-1}F\left(\alpha-\gamma+1, 1-\beta, \alpha-\beta+1, \frac{1}{1-z}\right).$

II*d*: $z^{1-\gamma}(1-z)^{\gamma-\beta-1}F\left(\beta-\gamma+1, 1-\alpha, \beta-\alpha+1, \frac{1}{1-z}\right).$

III*a*: $F(\alpha, \beta, \gamma, z).$
III*b*: $(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma, z).$
III*c*: $z^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z).$
III*d*: $z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}F(1-\alpha, 1-\beta, 2-\gamma, z).$

IV*a*: $F(\alpha, \beta, \alpha+\beta-\gamma+1, 1-z).$
IV*b*: $z^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, \alpha+\beta-\gamma+1, 1-z).$

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* For another method of attack yielding Kummer's solutions see Whittaker and Watson, 4th ed., 283–285.

IV*c*: $(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-z).$

IV*d*: $z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}F(1-\alpha, 1-\beta, \gamma-\alpha-\beta+1, 1-z).$

V*a*: $(1-z)^{-\alpha}F\left(\alpha, \gamma-\beta, \gamma, \dfrac{z}{z-1}\right).$

V*b*: $(1-z)^{-\beta}F\left(\beta, \gamma-\alpha, \gamma, \dfrac{z}{z-1}\right).$

V*c*: $z^{1-\gamma}(1-z)^{\gamma-\alpha-1}F\left(\alpha-\gamma+1, 1-\beta, 2-\gamma, \dfrac{z}{z-1}\right).$

V*d*: $z^{1-\gamma}(1-z)^{\gamma-\beta-1}F\left(\beta-\gamma+1, 1-\alpha, 2-\gamma, \dfrac{z}{z-1}\right).$

VI*a*: $z^{-\alpha}F\left(\alpha, \alpha-\gamma+1, \alpha+\beta-\gamma+1, \dfrac{z-1}{z}\right).$

VI*b*: $z^{-\beta}F\left(\beta, \beta-\gamma+1, \alpha+\beta-\gamma+1, \dfrac{z-1}{z}\right).$

VI*c*: $z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}F\left(1-\alpha, \gamma-\alpha, \gamma-\alpha-\beta+1, \dfrac{z-1}{z}\right).$

VI*d*: $z^{\beta-\gamma}(1-z)^{\gamma-\alpha-\beta}F\left(1-\beta, \gamma-\beta, \gamma-\alpha-\beta+1, \dfrac{z-1}{z}\right).$

We note in particular that the six cases III*a*, III*c*, IV*a*, IV*c*, I*a*, and I*b* give precisely those solutions whose presence is indicated in the Riemann symbol for the hypergeometric equation.

Among any three solutions which have a common region of definition there must exist a linear relation with constant coefficients. The determination of such constants is usually a delicate task and is certainly outside the scope of this book. Reference may be made to Forsyth on this matter.

We have not utilized the simplest devices available for the purpose of arriving at these twenty-four solutions. It does seem, however, that the method we have followed has intrinsic interest and importance apart from its success in obtaining the specific results for which it was used. That is, it may be that a clear conception of the method employed here will prove to be of more value than the mere possession of Kummer's* solutions.

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* Ernest Edward Kummer (1810–1893) was a German mathematician who made many important contributions to the subject. His main interest was the theory of numbers.

PROBLEM

In the hypergeometric equation put $z = 1 - \frac{1}{z_1}$, and then $w = z_1^\alpha w_1$, and thus verify the solution indicated in II*a*. Perform similar verifications in a few other cases.

52. Contiguous Functions. We touch very briefly the topic of contiguous functions. Gauss called the six functions $F(\alpha \pm 1, \beta, \gamma, z)$, $F(\alpha, \beta \pm 1, \gamma, z)$, $F(\alpha, \beta, \gamma \pm 1, z)$ contiguous to the function $w(z) = F(\alpha, \beta, \gamma, z)$. He proved that $w(z)$ and any two of its contiguous functions are connected by a linear relation with coefficients which are linear polynomials in z. There are, of course, $15 = \frac{6 \cdot 5}{1 \cdot 2}$ such relations. They are listed in Gauss, *Werke*, vol. 3, 130. We include one example:

$$\begin{aligned}\gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)z]F(\alpha, \beta, \gamma, z)& \\ + (\gamma - \alpha)(\gamma - \beta)zF(\alpha, \beta, \gamma + 1, z)& \\ = \gamma(\gamma - 1)(1 - z)F(\alpha, \beta, \gamma - 1, z)&.\end{aligned}$$

A much more complicated situation arises with the definition of contiguous functions as related to the Riemann-Papperitz equation (58), page 139. In Whittaker and Watson* there is a discussion of that definition of contiguous functions. Their work uses tools which we have not developed here. It is sufficient for us to note that Riemann proved that in such a case the original function (solution of the Riemann-Papperitz equation) and any two of its contiguous functions are connected by a linear relation with coefficients which are polynomials in z. This time the polynomial coefficients are not necessarily linear. Further, the original function has 30 contiguous functions and there are, therefore, $435 = \frac{30 \cdot 29}{1 \cdot 2}$ such linear relations! With this we drop the subject.

53. Special and Limiting Cases of the Hypergeometric Function. One of the reasons for interest in the hypergeometric function is the fact that so many known† functions may be expressed as special or limiting cases of that function. Also many relations, among known functions or others, may be written more compactly by means of the symbol $F(\alpha, \beta, \gamma, z)$. On occasion it may be that, when such a notation is used, our knowledge of certain properties of the hypergeometric function will suggest new results or simplifications of known developments.

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* See the 4th ed., 294–296.

† We are using the expression "known function" in the sense of a function which has received sufficient attention to have been given a name.

In the power series $F(\alpha, \beta, \gamma, z)$ the factor $(n + \alpha)$ occurs for the first time in the coefficient of z^{n+1}. Hence, if α is a negative integer $(-k)$, the series will terminate and will represent a polynomial of degree k. Thus, if k is a positive integer, the function $F(-k, \beta, \gamma, z)$ is a polynomial of degree k. We may represent many polynomials in this manner. An important special case is found when we choose $\beta = k + 1$, $\gamma = 1$, $z = \frac{1}{2}(1 - x)$. This choice yields what is known as a Legendre* polynomial or a zonal harmonic, a function of importance in applications as well as in pure mathematics. We write

$$P_k(x) = F\left(-k, k + 1, 1, \frac{1 - x}{2}\right).$$

Suppose now that we take $\beta = \gamma$. Then the series simplifies a great deal and we find

$$F(\alpha, \beta, \beta, z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1)}{n!} z^n.$$

But this is the familiar binomial expansion of $(1 - z)^{-\alpha}$. Hence $(1 - z)^{-\alpha} = F(\alpha, \beta, \beta, z)$.

Finally we list a few other results which the student should verify.

$$\begin{aligned}
&(a) && \log(1 + z) = zF(1, 1, 2, -z).\\
&(b) && \arcsin z = zF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, z^2).\\
&(c) && \arctan z = zF(\tfrac{1}{2}, 1, \tfrac{3}{2}, -z^2).\\
(66)\quad &(d) && \cos kz = F\left(\frac{k}{2}, -\frac{k}{2}, \frac{1}{2}, \sin^2 z\right).\\
&(e) && \cos z = \lim_{\alpha, \beta \to \infty} F\left(\alpha, \beta, \frac{1}{2}, -\frac{z^2}{4\alpha\beta}\right).\\
&(f) && \log\frac{1 + z}{1 - z} = 2zF(\tfrac{1}{2}, 1, \tfrac{3}{2}, z^2).\\
&(g) && e^z = \lim_{\alpha \to \infty} F\left(\alpha, \beta, \beta, \frac{z}{\alpha}\right).
\end{aligned}$$

---

* Adrien Marie Legendre (1752–1833) was a French mathematician whose researches have done much for both pure and applied mathematics.

There are lists of this type in many places. See, for example, Gauss, *op. cit.*, 127, Forsyth, and also the *Smithsonian Mathematical Formulae and Tables of Elliptic Functions.*

54. Legendre's Equation. It is worth noting how the important equation of Legendre fits into our scheme of things. Let us return to Riemann's symbol and obtain the equation* corresponding to

$$P\begin{pmatrix} -1 & 1 & \infty & \\ 0 & 0 & n+1 & z \\ 0 & 0 & -n & \end{pmatrix}.$$

We use equation (57), page 136, and in it put

$$\begin{array}{llll} z_1 = -1, & \alpha_{11} = 0, & \alpha_{12} = 0, & \alpha_{1\infty} = n+1, \\ z_2 = 1, & \alpha_{21} = 0, & \alpha_{22} = 0, & \alpha_{2\infty} = -n. \end{array}$$

The equation is found to be

$$\frac{d^2w}{dz^2} + \left[\frac{1}{z+1} + \frac{1}{z-1}\right]\frac{dw}{dz} - \frac{n(n+1)}{(z+1)(z-1)}\, w = 0,$$

or

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + n(n+1)w = 0,$$

which is the usual form for Legendre's equation. One solution of this equation, if n is a positive integer, is the frequently used Legendre polynomial of degree n. See Section 70 for more detail.

55. Two Special Results. First we obtain a simple result which is of a somewhat different type from our previous excursions into the properties of the hypergeometric function. Consider

$$\frac{d}{dz}F(\alpha, \beta, \gamma, z) = \sum_{n=1}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+n)} \frac{z^{n-1}}{(n-1)!}.$$

We may replace n by $(n+1)$ in the summation and obtain

$$\frac{d}{dz}F(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+1+n)} \frac{z^n}{n!}.$$

---

* In order to see how this equation may arise in practice see R. V. Churchill, *Fourier Series and Boundary Value Problems*, 1941, 193–201. One of the applications there is on the determination of the potential about a spherical surface. Churchill devotes a chapter to an elementary treatment of Legendre polynomials. For more extensive results and less elementary methods, see Whittaker and Watson, Chapter 15.

This is so similar to the hypergeometric function that we are tempted to try to identify it. It is easy to see that we may do so by the following rearrangement.

$$\frac{d}{dz} F(\alpha, \beta, \gamma, z) = \frac{\alpha\beta}{\gamma} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + 1 + n)\Gamma(\beta + 1 + n)\Gamma(\gamma + 1)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)\Gamma(\gamma + 1 + n)} \frac{z^n}{n!}.$$

Hence

$$\frac{d}{dz} F(\alpha, \beta, \gamma, z) = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1, z).$$

Next we state an important result which we need to use later. We omit the proof. The proof in Whittaker and Watson involves infinite products and a property of the gamma function which is not available in this course. The result is

$$(67) \qquad F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}.$$

MISCELLANEOUS PROBLEMS

1. Determine the region of convergence of the series $F(\alpha, \beta, \gamma, z)$.

2. Prove that

$$F(\alpha, \beta + 1, \gamma, z) - F(\alpha, \beta, \gamma, z) = \frac{\alpha z}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1, z).$$

3. Solve the equation

$$z(2 - z) \frac{d^2w}{dz^2} + 2(a - 2z) \frac{dw}{dz} - 2w = 0,$$

in which a is constant.

4. Show that you can solve any equation of the form

$$(1 - z^2) \frac{d^2w}{dz^2} + az \frac{dw}{dz} + bw = 0,$$

in which a and b are constants, by using the transformation $z^2 = x$ and your knowledge of the hypergeometric equation. In particular, obtain the general solution of the equation

$$9(1 - z^2) \frac{d^2w}{dz^2} - 2w = 0.$$

5. Show that, in general, the equation

$$z^2(k - z)^2 \frac{d^2w}{dz^2} + (az^2 + bz + c)w = 0,$$

in which a, b, c, and k are constants, can be solved with the aid of the hypergeometric equation.

CHAPTER VII

CONFLUENCE OF SINGULARITIES

56. Introductory Discussion. There is a procedure by which we can pass from an equation of Fuchsian type to one which has an irregular singular point. We may let two or more singularities come together in such a way that: (1) at least one of the corresponding exponents approaches infinity, and (2) there exists a limiting form of the corresponding differential equation. This process is called *confluence.*

It seems evident that we may occupy ourselves with either of two aims in applying confluence to an equation of Fuchsian type. We may attempt to get a familiar equation as a result of the confluence, or we may attempt to discover precisely what equations may be obtained from a specific equation by means of confluence. We shall take examples of each of these processes.

Before proceeding to the mode of attack which will be our main tool in the treatment of confluence, it may be worth our while to note at least one other viewpoint. Suppose that we wish to transform by confluence the hypergeometric equation into Bessel's equation* of index zero. We are naturally tempted first to try to find a way to pass from the hypergeometric series to the series for $J_0(z)$. That is, we wish to transform

$$F(\alpha, \beta, \gamma, z) =$$
$$1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)} \frac{z^n}{n!}$$

into

$$J_0(z) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{2^{2n}(n!)^2}.$$

At first glance we see that the choice $\gamma = 1$ will yield the desired factor $(n!)^2$ in the denominator. The factors $(-1)^n$ and 2^{2n} can easily be introduced by a change of independent variable. We have yet to cope with the $2n$ factors $\alpha, \alpha + 1, \cdots, \alpha + n - 1$, $\beta, \beta + 1, \cdots, \beta + n - 1$. If we replace z in the hypergeometric series by $-\dfrac{z^2}{4\alpha\beta}$, then an additional factor $\alpha^n\beta^n$ will appear in the denominator. Now all that remains to effect the desired elimination of α and β is to let them approach infinity. In detail we see that

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* See Section 66.

$$F\left(\alpha, \beta, 1, -\frac{z^2}{4\alpha\beta}\right) =$$

$$1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha + 1)\cdots(\alpha + n - 1)\beta(\beta + 1)\cdots(\beta + n - 1)}{n!} \frac{(-1)^n z^{2n}}{2^{2n}\alpha^n\beta^n n!}$$

$$= 1 +$$

$$\sum_{n=1}^{\infty} 1\cdot\left(1 + \frac{1}{\alpha}\right)\cdots\left(1 + \frac{n-1}{\alpha}\right)\cdot 1\cdot\left(1 + \frac{1}{\beta}\right)\cdots\left(1 + \frac{n-1}{\beta}\right)\frac{(-1)^n z^{2n}}{2^{2n}(n!)^2},$$

and hence that

$$\lim_{\alpha, \beta \to \infty} F\left(\alpha, \beta, 1, -\frac{z^2}{4\alpha\beta}\right) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{2^{2n}(n!)^2} = J_0(z).$$

Thus we are led to transform the hypergeometric equation as follows:

(*a*) Put $\gamma = 1$.

(*b*) Replace z by $-\dfrac{z^2}{4\alpha\beta}\cdot$

(*c*) Let $\alpha, \beta \to \infty$.

Actually these steps do lead to the desired result. Putting $\gamma = 1$ and replacing z by $-\dfrac{z^2}{4\alpha\beta}$ in the hypergeometric equation,

$$z(1 - z)\frac{d^2w}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{dw}{dz} - \alpha\beta w = 0,$$

leads to the equation

$$\left(1 + \frac{z^2}{4\alpha\beta}\right)\frac{d^2w}{dz^2} + \frac{1}{z}\left[1 + \frac{2\alpha + 2\beta + 1}{4\alpha\beta}z^2\right]\frac{dw}{dz} + w = 0,$$

a result which should be checked by the student. If, in the equation directly above, we let $\alpha, \beta \to \infty$, we arrive at Bessel's equation of index zero, namely,

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + w = 0.$$

There are unfortunately two serious objections to our adoption of the above procedure. First, the reason for our success in arriving at *this particular* equation, one of whose solutions is $J_0(z)$, is essentially governed by considerations we did not employ. To put it bluntly, we were lucky.* Secondly, the change of independent variable from z to $-\dfrac{z^2}{4\alpha\beta}$ will alter the number of singularities of the differential equation. This places the transformation outside the scope of our work.

Finally we note that, though neither of the objections raised above is insurmountable, they do indicate unnecessary complications. Therefore we shall turn to a process which may seem less natural but will prove much more satisfactory to us.

57. Another Mode of Attack. We shall now illustrate another method of attack. Let us consider the Fuchsian equation with only three singularities, those being located at $z = 0, c, \infty$. We shall permit the exponents at $z = c$ and $z = \infty$ to be polynomials in the parameter c. Then as we let $c \to \infty$ we shall have precisely the process we defined as confluence, provided that we make certain that the differential equation has a corresponding limiting form. This last condition can be shown to necessitate that the exponents at $z = c$ and $z = \infty$ be at most linear in c. We shall omit proof of that fact.

Thus we have decided to treat the equation indicated by

$$P\begin{pmatrix} 0 & c & \infty & \\ \alpha_{11} & \alpha_{12} + c\beta_{12} & \alpha_{1\infty} + c\beta_{1\infty} & z \\ \alpha_{21} & \alpha_{22} + c\beta_{22} & \alpha_{2\infty} + c\beta_{2\infty} & \end{pmatrix},$$

---

* As an example where the hidden reasons for our success above do not come to our rescue, consider the problem of transforming the hypergeometric equation into the equation $\dfrac{d^2w}{dz^2} - w = 0$. If the student applies the transformations suggested in (66*g*), page 160, he will arrive at a quite different equation, one of whose solutions is $w = e^z$.

in which the α's and β's are complex constants independent of c. Reference to equation (57), page 136, shows that the equation under consideration is

$$\frac{d^2w}{dz^2}+\left[\frac{1-\alpha_{11}-\alpha_{21}}{z}+\frac{1-\alpha_{12}-\alpha_{22}-c(\beta_{12}+\beta_{22})}{z-c}\right]\frac{dw}{dz}$$
$$+\left[\frac{\alpha_{11}\alpha_{21}}{z^2}+\frac{\alpha_{12}\alpha_{22}+c(\alpha_{12}\beta_{22}+\alpha_{22}\beta_{12})+c^2\beta_{12}\beta_{22}}{(z-c)^2}\right.$$
$$+\frac{\alpha_{1\infty}\alpha_{2\infty}-\alpha_{11}\alpha_{21}-\alpha_{12}\alpha_{22}+c(\alpha_{1\infty}\beta_{2\infty}+\alpha_{2\infty}\beta_{1\infty}-\alpha_{12}\beta_{22}-\alpha_{22}\beta_{12})}{z(z-c)}$$
$$\left.+\frac{c^2(\beta_{1\infty}\beta_{2\infty}-\beta_{12}\beta_{22})}{z(z-c)}\right]w=0.$$

It is easy to see that, if this equation is to have a limiting form as $c \to \infty$, then we must have

$$\beta_{1\infty}\beta_{2\infty}-\beta_{12}\beta_{22}=0. \tag{68}$$

The Fuchsian invariant for our differential equation has the value 1. Hence

$$\alpha_{11}+\alpha_{21}+\alpha_{12}+\alpha_{22}+\alpha_{1\infty}+\alpha_{2\infty}+c(\beta_{12}+\beta_{22}+\beta_{1\infty}+\beta_{2\infty})=1.$$

But this relation is to hold for all c and with the α's and β's independent of c. From this it follows at once that

$$\beta_{12}+\beta_{22}+\beta_{1\infty}+\beta_{2\infty}=0, \tag{69}$$

and

$$\alpha_{11}+\alpha_{21}+\alpha_{12}+\alpha_{22}+\alpha_{1\infty}+\alpha_{2\infty}=1. \tag{70}$$

Assuming that equations (68), (69), and (70) are satisfied, we may proceed to the limiting form of our differential equation,

$$\frac{d^2w}{dz^2}+\left[\frac{1-\alpha_{11}-\alpha_{21}}{z}+\beta_{12}+\beta_{22}\right]\frac{dw}{dz} \tag{71}$$
$$+\left[\frac{\alpha_{11}\alpha_{21}}{z^2}+\beta_{12}\beta_{22}-\frac{\alpha_{1\infty}\beta_{2\infty}+\alpha_{2\infty}\beta_{1\infty}-\alpha_{12}\beta_{22}-\alpha_{22}\beta_{12}}{z}\right]w=0,$$

in which, of course, the restrictions (68)–(70) still hold.

In a crude sense we have completed the solution of the problem of determining precisely what differential equations may be derived from the Fuchsian equation with three singularities by performing (*a*) to (*e*) as indicated.

(*a*) Keep one singularity fixed at $z = 0$.

(*b*) Hold fixed the exponents at $z = 0$.

(*c*) Put the other two singularities at $z = c$ and $z = \infty$.

(*d*) Permit the exponents at $z = c$ and $z = \infty$ to be polynomials in c.

(*e*) Let $c \to \infty$.

Our solution is incomplete in the sense that we have not analyzed it sufficiently to come to any reasonable comprehension of our result. We shall make some progress toward that comprehension in this and the next chapter. One major contribution toward a fuller understanding of (71) will be made when we obtain a comparatively simple canonical form into which any particular case of equation (71) may be thrown by " elementary " transformations on the dependent and independent variables.

58. Bessel's Equation. Before proceeding to a more detailed examination of

$$(71)\quad \frac{d^2w}{dz^2} + \left[\frac{1 - \alpha_{11} - \alpha_{21}}{z} + \beta_{12} + \beta_{22}\right]\frac{dw}{dz}$$

$$+ \left[\frac{\alpha_{11}\alpha_{21}}{z^2} + \beta_{12}\beta_{22} - \frac{\alpha_{1\infty}\beta_{2\infty} + \alpha_{2\infty}\beta_{1\infty} - \alpha_{12}\beta_{22} - \alpha_{22}\beta_{12}}{z}\right] w = 0,$$

let us return to the problem of obtaining Bessel's equation by confluence. In Section 56 we concerned ourselves with Bessel's equation of index 0. This time we shall obtain the equation of index n, including the equation of index 0 as a special case.

We wish to arrive at the equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{n^2}{z^2}\right) w = 0.$$

This equation* has a regular singular point at $z = 0$ with exponents n and $-n$ at that point, and it has an irregular singular point at $z = \infty$. We shall, of course, choose $\alpha_{11} = n$, $\alpha_{21} = -n$, and then by direct comparison of Bessel's equation with equation (71) set up a system of equations from which the remaining parameters are to be determined. This system of equations must include equations (68)-(70). The required system is:

---

* For a treatment of Bessel's equation, Bessel functions, and applications, see Churchill, *op. cit.*, Chapter 8. See also Whittaker and Watson, Chapter 17.

$$
\begin{aligned}
\beta_{1\infty}\beta_{2\infty} - \beta_{12}\beta_{22} &= 0, \\
\beta_{12} + \beta_{22} + \beta_{1\infty} + \beta_{2\infty} &= 0, \\
\alpha_{12} + \alpha_{22} + \alpha_{1\infty} + \alpha_{2\infty} &= 1, \\
\beta_{12} + \beta_{22} &= 0, \\
\beta_{12}\beta_{22} &= 1, \\
\alpha_{1\infty}\beta_{2\infty} + \alpha_{2\infty}\beta_{1\infty} - \alpha_{12}\beta_{22} - \alpha_{22}\beta_{12} &= 0.
\end{aligned} \tag{72}
$$

The solution of this system of equations is a matter of elementary algebra and is omitted here. There are four sets of solutions, each with two undetermined parameters, but all the results differ only in a trivial manner, such as an interchange of exponents at one singularity in equation (71). A solution is $\beta_{12} = i$, $\beta_{22} = -i$, $\beta_{1\infty} = i$, $\beta_{2\infty} = -i$, $\alpha_{1\infty} = \frac{1}{2}-\alpha_{22}$, $\alpha_{2\infty} = \frac{1}{2}-\alpha_{12}$, where $i = \sqrt{-1}$ and where α_{12} and α_{22} are arbitrary. We have proved that, if, in the equation defined by

$$
P\begin{pmatrix} 0 & c & \infty & \\ n & \alpha_{12}+ic & \frac{1}{2}-\alpha_{22}+ic & z \\ -n & \alpha_{22}-ic & \frac{1}{2}-\alpha_{12}-ic & \end{pmatrix},
$$

where α_{12} and α_{22} are arbitrary, we let $c \to \infty$, the result is Bessel's equation of order n. We have also shown that under certain conditions the above is the most general approach by confluence to Bessel's equation.

We wish to call attention to the fact that the problem of this section is not to be looked upon as an example of some mathematical parlor game, " Get one equation from the other." Such amusements appeal only to a small minority. Our interest is centered in the hope that, for example, known results on the Fuchsian equation may now suggest new results on Bessel's equation. In actual practice Bessel's equation has been studied much more in detail* than the Fuchsian equation, so that our remarks are to be taken to apply merely to the type of problem represented by this section.

To continue with the comments on this typical problem, we remark that it seems natural to construct results for the Fuchsian equation analogous to those we have discussed briefly for the hypergeometric equation and then to determine the limiting forms of those results in view of the processes of this section. We shall not do so in this book.

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* See, for example, the very extensive work, *Theory of Bessel Functions*, by G. N. Watson, Cambridge, 1922.

59. A Canonical Form: Whittaker's Equation. We return now to

$$(71)\quad \frac{d^2w}{dz^2}+\left[\frac{1-\alpha_{11}-\alpha_{21}}{z}+\beta_{12}+\beta_{22}\right]\frac{dw}{dz}$$

$$+\left[\frac{\alpha_{11}\alpha_{21}}{z^2}+\beta_{12}\beta_{22}-\frac{\alpha_{1\infty}\beta_{2\infty}+\alpha_{2\infty}\beta_{1\infty}-\alpha_{12}\beta_{22}-\alpha_{22}\beta_{12}}{z}\right]w=0,$$

in the hope of obtaining a better understanding of that equation. Let us obtain the invariant in the normal form of that equation. It may be found to be

$$I=A+\frac{B}{z}+\frac{C}{z^2},$$

where

$$\begin{aligned} A &= -\tfrac{1}{4}(\beta_{12}-\beta_{22})^2,\\ B &= \alpha_{12}\beta_{22}+\alpha_{22}\beta_{12}-\alpha_{1\infty}\beta_{2\infty}-\alpha_{2\infty}\beta_{1\infty}\\ &\qquad -\tfrac{1}{2}(1-\alpha_{11}-\alpha_{21})(\beta_{12}+\beta_{22}),\\ C &= \tfrac{1}{4}-\tfrac{1}{4}(\alpha_{11}-\alpha_{21})^2. \end{aligned}$$

We know now that equation (71) may be transformed into

$$\frac{d^2W}{dz^2}+\left(A+\frac{B}{z}+\frac{C}{z^2}\right)W=0,$$

by means of a transformation $w=f(z)W$, where $f(z)$ may be determined as in Section 3 and W is our new dependent variable. If we now replace the old independent variable z with a new one sz, s constant, the equation becomes

$$\frac{d^2W}{dz^2}+\left(As^2+\frac{Bs}{z}+\frac{C}{z^2}\right)W=0.$$

We may use s to eliminate one of the parameters A and B from our equation. In conformance with a standard notation we define s by

$$As^2=-\tfrac{1}{4},$$

and at the same time introduce two new parameters to replace Bs and C. These are k and m defined by

$$\begin{aligned} Bs &= k,\\ C &= \tfrac{1}{4}-m^2. \end{aligned}$$

Finally the equation which is our canonical form for equation (71) is seen to be

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right) W = 0. \tag{73}$$

This is called *Whittaker's confluent hypergeometric equation,* and it is the subject of our next chapter. With reference to its name, note that the Fuchsian equation from which we obtained equation (73) by confluence and transformation could itself easily have been obtained by elementary transformations from the hypergeometric equation.

PROBLEMS

1. Form the equation corresponding to the symbol

$$P\left\{ \begin{matrix} 0 & c & \infty & \\ \frac{1}{2} + m & c - k & -c & z \\ \frac{1}{2} - m & k & 0 & \end{matrix} \right\}.$$

Then let $c \to \infty$, and obtain an equation the normal form of which is Whittaker's equation.

2. Identify the α's and β's of Section 57 for the special case of the preceding problem. Then use equation (71) directly to arrive at Whittaker's equation.

CHAPTER VIII

WHITTAKER'S CONFLUENT HYPERGEOMETRIC EQUATION

60. Introduction. We have already met Whittaker's* equation,

$$\text{(73)} \qquad \frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) W = 0,$$

as a canonical form for equations derived by confluence from the general Fuchsian equation with three singular points. It is our intention to give here a brief introduction to the study of Whittaker's equation with a bare suggestion of some of the developments and problems related to it.

This equation has a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. At $z = 0$ the exponents of the equation are $(\frac{1}{2} + m)$ and $(\frac{1}{2} - m)$. Let us then attempt to obtain a solution " belonging to " the exponent $(\frac{1}{2} + m)$. We put

$$W = \sum_{n=0}^{\infty} a_n z^{n+m+(1/2)}.$$

Direct substitution yields

$$\sum_{n=0}^{\infty} (n+m+\tfrac{1}{2})(n+m-\tfrac{1}{2}) a_n z^{n+m-(3/2)} - \sum_{n=0}^{\infty} \tfrac{1}{4} a_n z^{n+m+(1/2)}$$
$$+ \sum_{n=0}^{\infty} k a_n z^{n+m-(1/2)} + \sum_{n=0}^{\infty} (\tfrac{1}{4} - m^2) a_n z^{n+m-(3/2)} = 0.$$

Proceeding in the usual manner, we find

$$\sum_{n=0}^{\infty} [(n+m)^2 - m^2] a_n z^{n+m-(3/2)} + \sum_{n=1}^{\infty} k a_{n-1} z^{n+m-(3/2)}$$
$$- \sum_{n=2}^{\infty} \tfrac{1}{4} a_{n-2} z^{n+m-(3/2)} = 0.$$

---

* E. T. Whittaker is a living British mathematician who has done a great deal in what is often called " classical analysis," of which the subject matter of this book is a part.

Next we write

$$[ka_0 + (1 + 2m)a_1]z^{m-(1/2)} + \sum_{n=2}^{\infty} [n(n + 2m)a_n + ka_{n-1} - \tfrac{1}{4}a_{n-2}]z^{n+m-(3/2)} = 0.$$

Now we see that the a_n are to be determined from the following equations:

$$(1 + 2m)a_1 + ka_0 = 0,$$

and, for $n \geqq 2$,

$$n(n + 2m)a_n + ka_{n-1} - \tfrac{1}{4}a_{n-2} = 0, \tag{74}$$

in which $a_0 \neq 0$ is arbitrary.

Again we have encountered a three-term recurrence relation.* In an attempt to avoid the difficulties apparent in using this type of relation, let us put $W = e^{cz}v$, where c is a constant. Then the factor e^{cz} will occur in each term of the transformed equation and can be removed. We hope to choose c so as to make the equation in v yield a two-term recurrence relation.

The transformation $W = e^{cz}v$ leads to the equation

$$\frac{d^2v}{dz^2} + 2c\frac{dv}{dz} + c^2v - \tfrac{1}{4}v + \frac{k}{z}v + \frac{\frac{1}{4} - m^2}{z^2}v = 0.$$

Consider a representative term, that involving z^p, in the series solution to be assumed for v. Corresponding to it the above six terms in the equation yield terms involving respectively z^{p-2}, z^{p-1}, z^p, z^p, z^{p-1}, z^{p-2}. It is the presence of the three different exponents which causes a recurrence relation with three a_n's in it. Let us then pick out the terms involving z^p and determine c so as to eliminate those terms. That is, we set $c^2 - \frac{1}{4} = 0$. Hence $c = \frac{1}{2}$ or $c = -\frac{1}{2}$.

If, then, we put $W = e^{z/2}v$ or $W = e^{-z/2}v$, we should be able to solve the recurrence relation needed in the series for v and thus write down relatively simple expressions which satisfy Whittaker's equation. This we proceed to do.

61. Solutions of Whittaker's Equation. Let us put $W = e^{z/2}v$. Then the equation for v is easily found to be

$$\frac{d^2v}{dz^2} + \frac{dv}{dz} + \left[\frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right]v = 0.$$

---

* See Section 37 for other examples.

Again the exponents at $z = 0$ are $(\frac{1}{2} + m)$ and $(\frac{1}{2} - m)$. Let us put

$$v = \sum_{n=0}^{\infty} b_n z^{n+m+(1/2)}.$$

We find at once that

$$\sum_{n=0}^{\infty} (n+m+\tfrac{1}{2})(n+m-\tfrac{1}{2})b_n z^{n+m-(3/2)} + \sum_{n=0}^{\infty} (n+m+\tfrac{1}{2})b_n z^{n+m-(1/2)}$$
$$+ \sum_{n=0}^{\infty} kb_n z^{n+m-(1/2)} + \sum_{n=0}^{\infty} (\tfrac{1}{4}-m^2)b_n z^{n+m-(3/2)} = 0.$$

Next we may write

$$\sum_{n=1}^{\infty} n(n + 2m)b_n z^{n+m-(3/2)}$$
$$+ \sum_{n=0}^{\infty} (n + m + k + \tfrac{1}{2})b_n z^{n+m-(1/2)} = 0,$$

or

$$\sum_{n=1}^{\infty} [n(n + 2m)b_n + (n + m + k - \tfrac{1}{2})b_{n-1}]z^{n+m-(3/2)} = 0.$$

Hence, for $n \geqq 1$,

$$b_n = -\frac{m + k - \frac{1}{2} + n}{n(2m + n)} b_{n-1},$$

with $b_0 \neq 0$ as an arbitrary constant.

The solution of this recurrence relation follows immediately by our customary method. We get

$$\begin{aligned} b_n &= -\frac{m + k - \frac{1}{2} + n}{n(2m + n)} b_{n-1} \\ &= (-1)^2 \frac{(m + k - \frac{1}{2} + n)(m + k - \frac{1}{2} + n - 1)}{n(n - 1)(2m + n)(2m + n - 1)} b_{n-2} = \cdots \\ &= (-1)^n \frac{(m+k-\frac{1}{2}+n)(m+k-\frac{1}{2}+n-1) \cdots (m+k-\frac{1}{2}+1)}{n(n-1) \cdots 2 \cdot 1(2m+n)(2m+n-1) \cdots (2m+1)} b_0, \end{aligned}$$

or

$$b_n = (-1)^n \frac{\Gamma(m + k + \frac{1}{2} + n)\Gamma(2m + 1)}{\Gamma(m + k + \frac{1}{2})\Gamma(2m + 1 + n)} \cdot \frac{b_0}{n!}.$$

We now choose $b_0 = 1$ and write the particular solution

$$v_1 = \frac{\Gamma(2m+1)}{\Gamma(m+k+\frac{1}{2})} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(m+k+\frac{1}{2}+n)}{\Gamma(2m+1+n)} \cdot \frac{z^{n+m+(1/2)}}{n!}.$$

Corresponding to the exponent $(\frac{1}{2} - m)$ we find another solution which is most easily obtained by replacing the parameter m in v_1 by $(-m)$. This second solution is

$$v_2 = \frac{\Gamma(-2m+1)}{\Gamma(-m+k+\frac{1}{2})} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(-m+k+\frac{1}{2}+n)}{\Gamma(-2m+1+n)} \cdot \frac{z^{n-m+(1/2)}}{n!}.$$

From v_1 and v_2 we obtain two solutions of Whittaker's equation, namely, $w_1 = e^{z/2}v_1$ and $w_2 = e^{z/2}v_2$.

Let us now consider the other substitution, $W = e^{-z/2}v$. It is easy to show that this leads to the equation

$$\frac{d^2v}{dz^2} - \frac{dv}{dz} + \left[\frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right] v = 0.$$

The exponents at $z = 0$ are again $(\frac{1}{2} + m)$ and $(\frac{1}{2} - m)$. We shall omit the details of the series solution, which parallels closely the work we have just finished. Note further that the differential equation directly above may be transformed into the equation at the beginning of this section by replacing z by $(-z)$ and k by $(-k)$. Hence we may write down solutions of the present equation by making those same substitutions in the expressions for v_1 and v_2. Thus we arrive at

$$v_3 = \frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{1}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(m-k+\frac{1}{2}+n)}{\Gamma(2m+1+n)} \cdot \frac{z^{n+m+(1/2)}}{n!},$$

$$v_4 = \frac{\Gamma(-2m+1)}{\Gamma(-m-k+\frac{1}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(-m-k+\frac{1}{2}+n)}{\Gamma(-2m+1+n)} \cdot \frac{z^{n-m+(1/2)}}{n!}.$$

We have obtained another pair of linearly independent solutions of Whittaker's equation. These are $w_3 = e^{-z/2}v_3$ and $w_4 = e^{-z/2}v_4$. The solutions w_3 and w_4 must, of course, be linearly dependent upon the solutions w_1 and w_2.

For purposes of reference we now write down the four solutions which we have obtained for Whittaker's equation.

(75) $w_1 =$

$$e^{z/2} \frac{\Gamma(2m+1)}{\Gamma(m+k+\frac{1}{2})} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(m+k+\frac{1}{2}+n)}{\Gamma(2m+1+n)} \frac{z^{n+m+(1/2)}}{n!}.$$

(76) $w_2 =$

$$e^{z/2}\frac{\Gamma(-2m+1)}{\Gamma(-m+k+\frac{1}{2})}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(-m+k+\frac{1}{2}+n)}{\Gamma(-2m+1+n)}\frac{z^{n-m+(1/2)}}{n!}.$$

(77) $w_3 =$

$$e^{-z/2}\frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(m-k+\frac{1}{2}+n)}{\Gamma(2m+1+n)}\frac{z^{n+m+(1/2)}}{n!}.$$

(78) $w_4 =$

$$e^{-z/2}\frac{\Gamma(-2m+1)}{\Gamma(-m-k+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(-m-k+\frac{1}{2}+n)}{\Gamma(-2m+1+n)}\frac{z^{n-m+(1/2)}}{n!}.$$

We have not considered the question of the region of convergence of the above power series. The student should show that each of these series is convergent for all finite values of z. Of course, certain exceptional values of the parameters m and k must be excluded. These the student can determine.

62. Kummer's First Formula. We know already that w_3 and w_4 are linearly dependent upon w_1 and w_2. More precisely, $w_1 = w_3$. This is called *Kummer's first formula.* It is equivalent to the identity

$$(79)\quad e^z\frac{\Gamma(2m+1)}{\Gamma(m+k+\frac{1}{2})}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(m+k+\frac{1}{2}+n)}{\Gamma(2m+1+n)}\frac{z^n}{n!}$$
$$=\frac{\Gamma(2m+1)}{\Gamma(m-k+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(m-k+\frac{1}{2}+n)}{\Gamma(2m+1+n)}\frac{z^n}{n!}.$$

We shall prove (79) by reducing the left member to the right member. We know that

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

Hence

$$e^z\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(2m+1)\Gamma(m+k+\frac{1}{2}+n)}{\Gamma(m+k+\frac{1}{2})\Gamma(2m+1+n)}\frac{z^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{s=0}^{n}(-1)^s\frac{\Gamma(2m+1)\Gamma(m+k+\frac{1}{2}+s)}{\Gamma(m+k+\frac{1}{2})\Gamma(2m+1+s)}\frac{1}{s!(n-s)!}z^n$$
$$=\sum_{n=0}^{\infty}\sum_{s=0}^{n}(-1)^s\frac{\Gamma(2m+1)\Gamma(m+k+\frac{1}{2}+s)}{\Gamma(m+k+\frac{1}{2})\Gamma(2m+1+s)}\frac{n!}{s!(n-s)!}\frac{z^n}{n!}.$$

Now

$$(-1)^s \frac{n!}{s!(n-s)!} = (-1)^s \frac{n(n-1)\cdots(n-s+1)}{s!}$$
$$= \frac{(-n)(-n+1)\cdots(-n+s-1)}{s!}.$$

Thus the coefficient of $\frac{z^n}{n!}$ in our infinite series may be written

$$\sum_{s=0}^{n} \frac{\Gamma(2m+1)\Gamma(m+k+\frac{1}{2}+s)}{\Gamma(m+k+\frac{1}{2})\Gamma(2m+1+s)} \frac{(-n)(-n+1)\cdots(-n+s-1)}{s!}$$

Since n is an integer, we see that this coefficient (compare with the formulas on page 146) is exactly

$$F(-n, m+k+\tfrac{1}{2}, 2m+1, 1).$$

Now is the time to use the result

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \tag{67}$$

which we stated on page 162. We have

$$F(-n, m+k+\tfrac{1}{2}, 2m+1, 1) = \frac{\Gamma(2m+1)\Gamma(m-k+\frac{1}{2}+n)}{\Gamma(2m+1+n)\Gamma(m-k+\frac{1}{2})},$$

so that, finally,

$$e^{z} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2m+1)\Gamma(m+k+\frac{1}{2}+n)}{\Gamma(m+k+\frac{1}{2})\Gamma(2m+1+n)} \frac{z^n}{n!}$$
$$= \sum_{n=0}^{\infty} F(-n, m+k+\tfrac{1}{2}, 2m+1, 1)\frac{z^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(2m+1)\Gamma(m-k+\frac{1}{2}+n)}{\Gamma(m-k+\frac{1}{2})\Gamma(2m+1+n)} \frac{z^n}{n!},$$

which completes the proof of Kummer's first formula, (79).

63. Whittaker's Confluent Hypergeometric Function. Whittaker finds it convenient to use as the fundamental confluent hypergeometric function a certain linear combination of w_3 and w_4, namely,

$$W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma(-m-k+\frac{1}{2})} w_3 + \frac{\Gamma(2m)}{\Gamma(m-k+\frac{1}{2})} w_4.$$

This definition is valid provided $2m$ is not integral.

We shall not go into any details concerning Whittaker's function. It appears that many known functions may be expressed simply in terms of Whittaker's function. Some such results are given in Whittaker and Watson. See, for example, pages 340–342 of the fourth edition.

Note that Whittaker and Watson have used the notation $M_{k,m}(z)$ for our w_3 and similar notations for w_1, w_2, and w_4. See also page 360, where the Bessel function $J_n(z)$ is related to our w_3. Finally, we remark that other cases,* not previously studied, of Whittaker's function prove to be of interest.

64. Solution of the Three-Term Recurrence Relation. The method used in proving Kummer's first formula suggests that we might be able to use the hypergeometric function to write w_1 explicitly as a power series in z and thus solve the three-term recurrence relation,

$$n(n+2m)a_n + ka_{n-1} - \tfrac{1}{4}a_{n-2} = 0, \tag{74}$$

in an indirect manner. The method of attack is dependent on the fact that $z^{-m-(1/2)}w_1$ has the value unity at $z = 0$ and that w_1 is a solution of Whittaker's equation. That is, w_1 is the particular solution using $a_0 = 1$ in the series

$$\sum_{n=0}^{\infty} a_n z^{n+m+(1/2)}$$

introduced on page 171, which led to the recurrence relation (74).

Recalling that

$$w_1 = e^{z/2}\frac{\Gamma(2m+1)}{\Gamma(m+k+\frac{1}{2})}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(m+k+\frac{1}{2}+n)}{\Gamma(2m+1+n)}\frac{z^{n+m+(1/2)}}{n!},$$

and that

$$e^{z/2} = \sum_{n=0}^{\infty}\frac{z^n}{2^n n!},$$

we see that we may write

$$w_1 = \sum_{n=0}^{\infty}\sum_{s=0}^{n}(-1)^s\frac{\Gamma(2m+1)\Gamma(m+k+\frac{1}{2}+s)}{\Gamma(m+k+\frac{1}{2})\Gamma(2m+1+s)}\frac{z^{n+m+(1/2)}}{2^{n-s}s!(n-s)!}.$$

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* See an article by Bateman, "The k-Function, a Particular Case of the Confluent Hypergeometric Function," *Transactions of the Amer. Math. Society*, vol. 33, 1931, 817–831.

Thus we have

$$a_n = \frac{1}{2^n}\sum_{s=0}^{n} (-1)^s \frac{\Gamma(2m+1)\Gamma(m+k+\frac{1}{2}+s)}{\Gamma(m+k+\frac{1}{2})\,\Gamma(2m+1+s)}\,\frac{2^s}{s!(n-s)!}.$$

Next we write

$$\begin{aligned} a_n &= \frac{1}{2^n n!}\sum_{s=0}^{n} (-1)^s \frac{\Gamma(2m+1)\Gamma(m+k+\frac{1}{2}+s)}{\Gamma(m+k+\frac{1}{2})\Gamma(2m+1+s)}\,\frac{n!}{(n-s)!}\,\frac{2^s}{s!} \\ &= \frac{1}{2^n n!}\sum_{s=0}^{n} \frac{\Gamma(2m+1)\Gamma(m+k+\frac{1}{2}+s)}{\Gamma(m+k+\frac{1}{2})\Gamma(2m+1+s)} \times \\ &\qquad\qquad \frac{(-n)(-n+1)\cdots(-n+s-1)}{1}\,\frac{2^s}{s!}. \end{aligned}$$

Finally, since n is a non-negative integer,

$$a_n = \frac{1}{2^n n!}\,F(-n, m+k+\tfrac{1}{2}, 2m+1, 2). \tag{80}$$

We have proved that

$$w_1 = \sum_{n=0}^{\infty} \frac{F(-n, m+k+\frac{1}{2}, 2m+1, 2)}{2^n n!}\,z^{n+m+(1/2)}, \tag{81}$$

or, in other words, that the a_n of (80) satisfies, for $n \geqq 3$, the recurrence relation,

$$n(n+2m)a_n + ka_{n-1} - \tfrac{1}{4}a_{n-2} = 0. \tag{74}$$

MISCELLANEOUS PROBLEMS

1. Show that

$$w_3 = z^{m+(1/2)}e^{-z/2} \lim_{\rho\to\infty} F\left(m-k+\tfrac{1}{2}, m-k+\tfrac{1}{2}+\rho, 2m+1, \frac{z}{\rho}\right).$$

2. Show that by a proper change of variables the equation

$$(a_2 + b_2 z)\frac{d^2w}{dz^2} + (a_1 + b_1 z)\frac{dw}{dz} + (a_0 + b_0 z)w = 0$$

can be transformed into

$$t\frac{d^2w}{dt^2} + (\gamma - t)\frac{dw}{dt} - \alpha w = 0,$$

and that the latter equation is derived from the equation for $F(\alpha, \beta, \gamma, z)$ by putting $z = \frac{t}{\beta}$ and letting $\beta \to \infty$.

Note: The second equation in Problem 2 is sometimes called the confluent hypergeometric equation. A good list of formulas and relations involving it is to be found in the *Smithsonian Mathematical Formulae and Tables of Elliptic Functions,* 185–187. See also Section 73 below.

3. Transform

$$4z^2 \frac{d^2w}{dz^2} + 16z \frac{dw}{dz} - (9z^2 - 4z - 9)w = 0.$$

into Whittaker's equation, and thus write down the solution of the original equation.

4. Solve by the method of Problem 3:

$$16z^2 \frac{d^2w}{dz^2} + 16z^2 \frac{dw}{dz} + 3w = 0.$$

5. Use $w = zw_1$ and $z = \frac{1}{\zeta}$ to solve the equation

$$16z^4 \frac{d^2w}{dz^2} - (5z^2 - 8z + 4)w = 0.$$

CHAPTER IX

SOME CLASSICAL EQUATIONS

65. Introduction. In this chapter we shall touch upon several ordinary linear differential equations which have been studied extensively in mathematical literature. These equations play important roles in both pure and applied mathematics. They have furnished us with the tools needed for the solution of many problems in engineering, physics, and statistics. The detailed study of these equations has suggested to pure mathematicians the formulation of, and even methods of attack upon, more general problems and theories.

It is beyond the scope of this book to discuss the many interesting properties of these special equations. For further study the reader may turn to the references included in this chapter.

We shall state solutions the derivations of which are left as problems for the student. These are for the most part direct applications of the technique developed in Chapter IV. At times we obtain very simple properties of these solutions and present others to be proved by the student. It should be understood that the properties included here are far too few and special to be considered even as representative of the theories of the special functions introduced in this chapter.

66. Bessel's Equation. We have several times used as examples or problems special cases of Bessel's equation,

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - n^2)\,w = 0. \tag{82}$$

One solution of this equation is

$$w = J_n(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\Gamma(k+n+1)} \left(\frac{z}{2}\right)^{2k+n}.$$

If n is not an integer, then $J_{-n}(z)$ is another solution, linearly independent of $J_n(z)$. If n is integral, the exponents of our equation at the regular singularity $z = 0$ differ by an integer. If

n is a non-negative integer, a second solution* linearly independent of $J_n(z)$ is

$$Y_n(z) = J_n(z)\log z - \frac{1}{2}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}$$
$$-\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\,\frac{H_k + H_{k+n}}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k+n}.$$

There is no need to consider negative integral values of n since that parameter enters equation (82) only in the term n^2w. Thus, if one exponent at $z = 0$ is a negative integer, the other is a positive integer and can be chosen as our n.

We shall obtain one of the many recurrence formulas satisfied by $J_n(z)$. Consider $z[J_{n-1}(z) + J_{n+1}(z)]$. From the definition of $J_n(z)$ we find that

$$\begin{aligned} z[J_{n-1}(z) + J_{n+1}(z)] &= \sum_{k=0}^{\infty}(-1)^k\,\frac{z}{k!\Gamma(k+n)}\left(\frac{z}{2}\right)^{2k+n-1} \\ &\quad + \sum_{k=0}^{\infty}(-1)^k\,\frac{z}{k!\Gamma(k+n+2)}\left(\frac{z}{2}\right)^{2k+n+1} \\ &= 2\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(k+n)}\left(\frac{z}{2}\right)^{2k+n} \\ &\quad + 2\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(k+n+2)}\left(\frac{z}{2}\right)^{2k+n+2}. \end{aligned}$$

Now a shift of index in the last summation permits us to write

$$\begin{aligned} z[J_{n-1}(z) + J_{n+1}(z)] &= 2\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(k+n)}\left(\frac{z}{2}\right)^{2k+n} \\ &\quad + 2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{(k-1)!\Gamma(k+n+1)}\left(\frac{z}{2}\right)^{2k+n}. \end{aligned}$$

But

$$\Gamma(k+n+1) = (k+n)\Gamma(k+n) \quad\text{and}\quad k! = k\cdot(k-1)!.$$

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* The $Y_n(z)$ used here is sometimes called Neumann's Bessel function of the second kind of index n.

Hence,

$$\begin{aligned} z[J_{n-1}(z) + J_{n+1}(z)] &= 2\sum_{k=0}^{\infty} (-1)^k \frac{k+n}{k!\Gamma(k+n+1)}\left(\frac{z}{2}\right)^{2k+n} \\ &\quad + 2\sum_{k=1}^{\infty} (-1)^{k-1} \frac{k}{k!\Gamma(k+n+1)}\left(\frac{z}{2}\right)^{2k+n} \\ &= \frac{2n}{\Gamma(n+1)}\left(\frac{z}{2}\right)^{n} + 2\sum_{k=1}^{\infty} (-1)^k \frac{n}{k!\Gamma(k+n+1)}\left(\frac{z}{2}\right)^{2k+n} \\ &= 2n\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\Gamma(k+n+1)}\left(\frac{z}{2}\right)^{2k+n}. \end{aligned}$$

Thus we have shown, for n not a negative integer, that

$$z[J_{n-1}(z) + J_{n+1}(z)] = 2nJ_n(z).$$

This is the recurrence formula which we desired. The $Y_n(z)$ have the same recurrence formula.

REFERENCES

WHITTAKER and WATSON, Chapter 17.

GRAY, A., MATHEWS, G. B., and MACROBERT, T. M. *A Treatise on Bessel Functions*, 1922.

WATSON, G. N., *Theory of Bessel Functions*, 1922.

CHURCHILL, R. V., *Fourier Series and Boundary Value Problems*, 1941, Chapter 8.

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PROBLEMS

1. Obtain the formula

$$\frac{d}{dz}[z^n J_n(z)] = z^n J_{n-1}(z).$$

2. Obtain the formula

$$J_{n-1}(z) - J_{n+1}(z) = 2\frac{d}{dz}J_n(z).$$

67. The Modified Bessel Equation. Consider the equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2 + n^2)w = 0, \tag{83}$$

obtained from Bessel's equation by replacing z by iz. It is easy to show that one solution of (83) is

$$w = I_n(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+n+1)} \left(\frac{z}{2}\right)^{2k+n}.$$

This function is called the modified Bessel function of the first kind of index n. The corresponding function of the second kind is often denoted $K_n(z)$.

REFERENCES: The first three references of the preceding section.

PROBLEMS

1. Show that

$$I_n(z) = i^{-n}J_n(iz).$$

2. Obtain a second solution of equation (83): (*a*) if n is non-integral; (*b*) if n is integral.

3. Show that

$$z[I_{n-1}(z) - I_{n+1}(z)] = 2nI_n(z),$$

both by using the result of Problem 1 and directly from the power series for $I_n(z)$.

4. Find other recurrence relations for $I_n(z)$. *Hint:* See the problems of Section 66.

68. The Bessel-Clifford Equation. The equation

$$z\frac{d^2w}{dz^2} + (\nu + 1)\frac{dw}{dz} + w = 0 \tag{84}$$

may be reduced to Bessel's equation by means of the transformations

$$w_1 = z^{\nu/2}w, \qquad z_1 = 2z^{1/2}.$$

The student should perform the indicated transformations and then write the solution of equation (84) in terms of Bessel functions.

69. Kelvin's Ber and Bei Functions and Their Generalizations. Lord Kelvin* introduced a terminology for the real and imaginary parts of the modified Bessel functions $I_0(x\sqrt{i})$ and $K_0(x\sqrt{i})$, where x is a real variable. See Section 67 for the definitions of $I_n(z)$ and $K_n(z)$ and for the power series for $I_n(z)$.

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* Sir William Thomson, Lord Kelvin, (1824–1907), a British mathematical physicist, whose work in applied mathematics is particularly well known.

Whitehead generalized Kelvin's definition to introduce the real and imaginary parts of $I_n(x\sqrt{i})$ and $K_n(x\sqrt{i})$. We use that same concept. Let x be a real variable, and then put†

$$\begin{aligned} i^n I_n(x\sqrt{i}) &= \operatorname{ber}_n(x) + i\operatorname{bei}_n(x), \\ i^n K_n(x\sqrt{i}) &= \operatorname{ker}_n(x) + i\operatorname{kei}_n(x), \end{aligned} \tag{85}$$

where $\operatorname{ber}_n x$, $\operatorname{bei}_n x$, etc., are real functions of the real variable x. The constant factor i^n has no effect on the differential equation satisfied by these functions.

If we return to

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2 + n^2)w = 0, \tag{83}$$

and in it put $z = x\sqrt{i}$, we find that

$$x^2\frac{d^2w}{dx^2} + x\frac{dw}{dx} - (ix^2 + n^2)w = 0. \tag{86}$$

We next separate w, which stands for either $I_n(x\sqrt{i})$ or $K_n(x\sqrt{i})$, into its real and imaginary parts. For simplicity we write $w(x) = H(x) + iM(x)$, in which either $H(x)$ denotes $\operatorname{ber}_n x$ and $M(x)$ denotes $\operatorname{bei}_n(x)$ or $H(x)$ denotes $\operatorname{ker}_n x$ and $M(x)$ denotes $\operatorname{kei}_n x$. We also let primes stand for differentiation with respect to x.

From equation (86) we have

$$\begin{aligned} x^2H'' + ix^2M'' + xH' + ixM' - ix^2H + x^2M& \\ - n^2H - in^2M &= 0. \end{aligned}$$

If we assume that n is real, we may separate the real and imaginary parts of this equation and arrive at the two results

$$\begin{aligned} x^2H'' + xH' + x^2M - n^2H &= 0, \\ x^2M'' + xM' - x^2H - n^2M &= 0. \end{aligned}$$

From the former equation we find that

$$M = -H'' - x^{-1}H' + n^2x^{-2}H.$$

Direct substitution of this expression for M into the equation

$$x^2M'' + xM' - x^2H - n^2M = 0$$

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† The ber is supposed to suggest "*B*essel, *r*eal part," and bei is to suggest "*B*essel, *i*maginary part."

yields, with some rearrangement,

$$(87)\quad x^4\frac{d^4H}{dx^4}+2x^3\frac{d^3H}{dx^3}-(1+2n^2)x^2\frac{d^2H}{dx^2}+(1+2n^2)x\frac{dH}{dx} + [x^4+n^2(n^2-4)]H=0.$$

We know that $\mathrm{ber}_n\, x$ and $\mathrm{ker}_n\, x$ are solutions of equation (87). It is easy to show that the equation for $M(x)$ is again equation (87) and therefore that $\mathrm{bei}_n\, x$ and $\mathrm{kei}_n\, x$ are also solutions of that equation. Thus the general solution of equation (87) is

$$H = c_1\,\mathrm{ber}_n\, x + c_2\,\mathrm{bei}_n\, x + c_3\,\mathrm{ker}_n\, x + c_4\,\mathrm{kei}_n\, x,$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

PROBLEMS

1. Write down the special case $n = 2$ of equation (87). Then subject the resultant equation to the transformations

$$H=\sigma^{-1}\psi \qquad \text{and} \qquad 4\sigma=x^2.$$

Ans. $\sigma^2\dfrac{d^4\psi}{d\sigma^4}+\psi=0.$

2. Solve equation (87), for Kelvin's case $n = 0$, in the neighborhood of the regular singular point $z = 0$. Note the important extension of the methods, definitions, and results of Chapter IV implied in this problem.

70. Legendre's Equation. We have already (Section 54, page 161) met Legendre's equation

$$(88)\qquad (1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+n(n+1)w=0.$$

We are particularly interested in non-negative integral n. For such a value of n one solution of equation (88) is the Legendre polynomial,*

$$P_n(z)=\sum_{k=0}^{[n/2]}(-1)^k\frac{(2n-2k)!}{2^nk!(n-k)!(n-2k)!}z^{n-2k},$$

where $\left[\dfrac{n}{2}\right]$ is the greatest integer in $\dfrac{n}{2}$. A solution linearly independent of $P_n(z)$ is

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* For examples of the use of Legendre polynomials in problems in elasticity, see S. Timoshenko, *Theory of Elasticity*, 1934, 313–318.

$$Q_n(z) = \tfrac{1}{2}P_n(z) \log \frac{z+1}{z-1} + \text{a power series in } z,$$

and should be obtained by the student.

Corresponding solutions when n is not a positive integer are called Legendre functions of the first kind and of the second kind. These may be obtained by the usual methods.

The equation

$$(89) \quad (1 - z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left[n(n+1) - \frac{m^2}{1-z^2}\right]w = 0$$

is often called the associated Legendre equation, and two of its solutions are denoted by $P_n^m(z)$ and $Q_n^m(z)$.

REFERENCES

WHITTAKER and WATSON, Chapter 15.
CHURCHILL, R. V., *op. cit.*, Chapter 9.
JACKSON, D., *op. cit.*, Chapter 2.

PROBLEMS

1. Prove the recurrence formula

$$(n+1)P_{n+1}(z) + nP_{n-1}(z) = (2n+1)zP_n(z).$$

2. Prove that

$$(2n+1)P_n(z) = \frac{d}{dz}[P_{n+1}(z) - P_{n-1}(z)].$$

3. Prove that for integral n

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n.$$

4. Use the result of Problem 3 to prove the formula of Problem 2.

5. Use the result of Problem 3 to prove the important orthogonality property,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, \quad \text{if} \quad m \neq n.$$

Hint: Use integration by parts repeatedly.

6. Prove that, for n integral,

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}\cdot$$

7. Prove for integral n the properties:

(*a*) $P_n(-z) = (-1)^n P_n(z)$.

(*b*) $P_n(1) = 1$.

(*c*) $P_{2n+1}(0) = 0$.

(*d*) $P_{2n}(0) = (-1)^n \dfrac{(2n)!}{2^{2n}(n!)^2}$.

8. Prove that, for m a positive integer, the function

$$w = P_n^m(z) = (1 - z^2)^{m/2} \frac{d^m}{dz^m} P_n(z)$$

is a solution of equation (89), directly above this set of problems.

71. Hermite Polynomials.

The differential equation,

$$\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + 2nw = 0, \tag{90}$$

has, for n a non-negative integer, the polynomial solution

$$H_n(z) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(2z)^{n-2k}}{k!(n-2k)!}.$$

The polynomial $H_n(z)$ is called the Hermite polynomial. It plays a role in certain problems in statistics.

We use the Hermite polynomials here to introduce an important concept, that of a *generating function.* Consider the expansion of $\exp(2zu - u^2)$ into a power series in u. This may be accomplished by multiplication of series, since

$$\exp(2zu - u^2) = \exp(2zu) \cdot \exp(-u^2).$$

Thus

$$\exp(2zu - u^2) = \left(\sum_{n=0}^{\infty} \frac{2^n z^n u^n}{n!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{n!}\right).$$

If we wish to obtain the coefficient of u^n in this product, we need to use some such device as that introduced on pages 76–77 for the multiplication of the series for $\exp(z)$ and $\cos z$. One thing is evident: the coefficient of u^n will be a polynomial in z. We omit the actual multiplication and merely state the result,

$$\exp(2zu - u^2) = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} u^n, \tag{91}$$

where the $H_n(z)$ is the Hermite polynomial. Equation (91) may, of course, be used to define the Hermite polynomials. In that event it is called a generating function for those polynomials; that is, expanding the function $\exp(2zu - u^2)$ into a power series in u generates the $H_n(z)$.

The generating function is often useful in deriving properties of the polynomials. Of course, all properties of $H_n(z)$ must follow directly or otherwise from equation (91), but for some of them the differential equation may prove to be a better starting point. In order to give one simple example of the use of the generating function, let us differentiate equation (91) with respect to z. We find that

$$2u \exp(2zu - u^2) = \sum_{n=1}^{\infty} \frac{H_n'(z)}{n!} u^n,$$

where the first term on the right of equation (91) dropped out under differentiation because $H_0(z) = 1$. From the above equation we have

$$\begin{aligned} 2 \exp(2zu - u^2) &= \sum_{n=1}^{\infty} \frac{H_n'(z)}{n!} u^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{H_{n+1}'(z)}{(n+1)!} u^n. \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} \frac{2H_n(z)}{n!} u^n = \sum_{n=0}^{\infty} \frac{H_{n+1}'(z)}{(n+1)!} u^n$$

from which $H_{n+1}'(z) = 2(n+1)H_n(z)$, or as it is more often written

$$H_n'(z) = 2nH_{n-1}(z).$$

REFERENCES

SZEGÖ, G., *Orthogonal Polynomials*, 1939.
JACKSON, D., *op. cit.*, Chapter 9.

PROBLEMS

1. Prove that

$$H_n(z) = 2zH_{n-1}(z) - H_{n-1}'(z).$$

2. From the result of Problem 1 and the relation $H_n'(z) = 2nH_{n-1}(z)$ prove that

$$H_n(z) = 2zH_{n-1}(z) - 2(n-1)H_{n-2}(z).$$

3. Assuming that

$$e^{-x^2}H_n(x) = (-1)^n \frac{d^n}{dx^n} e^{-x^2},$$

a property which can be deduced directly from the generating function, prove that

$$\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)\,dx = 0, \quad \text{if } m \neq n.$$

This is equivalent to the statement that " the Hermite polynomials are orthogonal with respect to the weight function $\exp(-x^2)$ in the interval $-\infty$ to $+\infty$."

4. Prove that

$$\int_{-\infty}^{\infty} e^{-x^2}H_n^2(x)\,dx = \sqrt{\pi}\, 2^n n!.$$

72. Laguerre Polynomials. If n is a non-negative integer, the equation

$$z\frac{d^2w}{dz^2} + (1 - z)\frac{dw}{dz} + nw = 0 \tag{92}$$

has a polynomial solution

$$w = L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\frac{z^k}{k!},$$

called a Laguerre polynomial. This polynomial enters in the study of statistics, heat conduction, and elsewhere.

There is an equation,

$$z\frac{d^2w}{dz^2} + (\alpha + 1 - z)\frac{dw}{dz} + nw = 0, \tag{93}$$

which also has a polynomial solution

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k}\frac{z^k}{k!},$$

sometimes also called the Laguerre polynomial, or the generalized Laguerre polynomial. Note that, if α is not an integer, we use the formal equivalent of the binomial coefficient in terms of the gamma function,

$$\binom{s}{t} = \frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)}.$$

Note also that $L_n(z) = L_n^{(0)}(z)$.

We shall restrict our discussion and problems to the simpler function $L_n(z)$. There is a generating function for $L_n(z)$ as indicated in the identity

$$e^u J_0(2\sqrt{zu}) = \sum_{n=0}^{\infty} \frac{L_n(z)}{n!} u^n.$$

Once more, much may be found directly from the generating function, but this time a reasonable knowledge of Bessel functions is needed to facilitate any such procedure.

There is an elementary generating function as indicated in

$$\frac{1}{1-u} \exp\left(-\frac{zu}{1-u}\right) = \sum_{n=0}^{\infty} L_n(z) u^n,$$

but the expansion of the left member is less simple to obtain than that of the other generating function above. Note the different forms of the right members in these two cases.

REFERENCES

SZEGÖ, G., *op. cit.*
JACKSON, D., *op. cit.*, Chapter 10.

PROBLEMS

1. Transform equation (92) into the form

$$\frac{d^2w}{dz^2} + \left[-\frac{1}{4} + \frac{n + \frac{1}{2}}{z} + \frac{\frac{1}{4}}{z^2}\right] w = 0,$$

and thus relate the polynomials of this section to Whittaker's confluent hypergeometric functions.

2. Assuming the relation

$$e^{-x} L_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (e^{-x} x^n),$$

prove the orthogonality relation

$$\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = 0, \quad \text{if } m \neq n.$$

3. Evaluate

$$\int_0^{\infty} e^{-x} L_n^2(x)\, dx.$$

73. The Pochhammer-Barnes Confluent Hypergeometric Equation.

In Chapter VIII we met the equation

$$z \frac{d^2w}{dz^2} + (\gamma - z) \frac{dw}{dz} - \alpha w = 0. \tag{94}$$

Equation (94) may be obtained from

$$\lim_{\beta\to\infty} P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & \dfrac{z}{\beta} \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{pmatrix},$$

and is therefore a confluent form of the hypergeometric equation. It is sometimes called the Pochhammer-Barnes confluent hypergeometric equation.

One solution of equation (94) may be written

$$\begin{aligned} w = M(\alpha, \gamma, z) &= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}\frac{z^n}{n!} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)}\frac{z^n}{n!}. \end{aligned}$$

Indeed, barring integral values of γ, the general solution of equation (94) is

$$w = AM(\alpha, \gamma, z) + Bz^{1-\gamma}M(\alpha-\gamma+1, 2-\gamma, z).$$

The theory of equation (94) and its solutions parallels that of the hypergeometric equation and functions to a certain extent. The existence of an irregular singular point at $z = \infty$ for equation (94) naturally places some bounds on this parallelism.

REFERENCE: *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, 1922, 185–188.

PROBLEMS

1. Identify the Laguerre polynomial $L_n(x)$ as a special case of the Pochhammer-Barnes confluent hypergeometric function $M(\alpha, \gamma, z)$.

2. Prove that

$$\frac{d}{dz}M(\alpha, \gamma, z) = \frac{\alpha}{\gamma}M(\alpha+1, \gamma+1, z).$$

3. Prove that

$$(1-\alpha)\int_0^z M(\alpha, \gamma, x)\,dx = (1-\gamma)M(\alpha-1, \gamma-1, z) + \gamma - 1.$$

4. Obtain equation (94) by confluence from the equation of Fuchsian type with three singular points, using the method of Section 57, page 165.

5. Put equation (94) into the form of Whittaker's equation, identify k and m, and write solutions of (94) with the aid of the results in (75)–(78); see page 174.

CHAPTER X

APPLICATIONS

74. Introduction. We shall close with a brief look at applications which lead to the study of some of the differential equations in the preceding chapters. It would appear pointless for us to deal with applications, no matter how interesting, unless they involve differential equations in the solution of which we can make use of the material in this book. Unfortunately, without appreciably increasing the prerequisites for the reader, we cannot attempt any derivation of equations or detailed discussion of these applications.

We do not intend to imply that the list of applications here is in any sense a complete one, or even completely representative. There are many fields in which ordinary differential equations are useful but which are not even mentioned here.

75. A Detail in the Design of Large Pipes. A pipe (penstock) of large diameter, such as the 30-foot-diameter ones being used in connection with Boulder Dam, may be strengthened with a device known as a stiffener ring.* The stiffener ring protrudes from the penstock so abruptly that it is wise to tone down the stresses in its vicinity by introducing what is called a fillet insert. This fillet insert brings to the design problem the need for a stress study for a circular pipe with a shell of linearly varying thickness.

The problem of determining radial displacement in the fillet insert due to internal water pressure may be reduced to the problem of solving the equation

$$z^2 \frac{d^4w}{dz^4} + 6z \frac{d^3w}{dz^3} + 6 \frac{d^2w}{dz^2} + w = 0. \tag{95}$$

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* See the Boulder Canyon Project Final Reports, Part V, Technical Investigations, *Bulletin 5*, *Penstock Analysis and Stiffener Design*, United States Bureau of Reclamation, Denver, 1940. The equivalent of our equation (95) appears on page 97 of that report. See also S. Timoshenko, *Theory of Plates and Shells*, 1940, 414, where a cylindrical tank with walls of uniformly varying thickness is studied.

In this equation w is proportional to the desired radial displacement and z is proportional to distance measured along an inside element of the pipe shell from a certain convenient point.

It is worth noting that we need the complete solution of equation (95) in order to use the arbitrary constants in performing the mathematical equivalent of hooking the fillet insert onto the pipe shell on the one side and onto the stiffener ring on the other side. This, of course, implies that we have at hand also the complete solutions for the pipe-shell and stiffener-ring problems. Those problems, as ordinarily treated, lead to differential equations whose solutions are very simple elementary functions and hold no particular interest for us here.

Equation (95) is, except for its order, of a type with which we are thoroughly familiar, and its solution is left for the problems.

PROBLEMS

1. Solve equation (95) completely in the vicinity of $z = 0$. See Problem 2 for a check.

2. Put $n = 1$ in equation (87), page 185, and then transform the result,

$$x^4 \frac{d^4H}{dx^4} + 2x^3 \frac{d^3H}{dx^3} - 3x^2 \frac{d^2H}{dx^2} + 3x \frac{dH}{dx} + (x^4 - 3)H = 0,$$

using in order the transformations $x^2 = 4z$ and $H = z^{1/2}w$. From your result write the complete solution of equation (95) in terms of generalized Kelvin functions.

Ans. $$w = z^{-1/2}[c_1 \operatorname{ber}_1 (2\sqrt{z}) + c_2 \operatorname{bei}_1 (2\sqrt{z}) + c_3 \operatorname{ker}_1 (2\sqrt{z}) + c_4 \operatorname{kei}_1 (2\sqrt{z})].$$

76. Deformation of Circular Plates. Timoshenko, in his *Theory of Plates and Shells*, 1940, 282–287, has discussed the deformation due to a uniform load on a circular plate of varying thickness. Let

r = distance from the center of the plate.
a = radius of the plate.
h_0 = thickness of the plate at its center.
h = thickness of the plate at a distance r from the center.
φ = angle between the normal to the deformed surface of the plate and the normal to that surface at the center of the plate.
ν = Poisson's ratio, a pure number characteristic of the material and physically restricted in range, $0 \leqq \nu < 0.5$.

Now put $x = \frac{r}{a}$ and $y = \frac{h}{h_0}$, and note that we shall be interested in the range $0 \leqq x \leqq 1$ and in functions $y(x)$ such that $y(x) > 0$ and $y(0) = 1$.

It is shown in Timoshenko, *loc. cit.*, that under sufficient restrictions of symmetry the problem at hand involves solving the equation*

$$(96) \qquad \frac{d^2\varphi}{dx^2} + \left(\frac{1}{x} + \frac{d}{dx}\log y^3\right)\frac{d\varphi}{dx} - \left(\frac{1}{x^2} - \frac{\nu}{x}\frac{d}{dx}\log y^3\right)\varphi = 0.$$

The function $y(x)$ is to be chosen so as to describe as closely as possible the varying thickness of the plate, always with the restrictions $y(0) = 1$, and $y(x) > 0$ for $0 \leqq x \leqq 1$.

Timoshenko considers in some detail the case $y = \exp\left(\frac{-\beta x^2}{6}\right)$, where β may be either a positive or a negative constant. This choice for y leads at once to the equation

$$(97) \qquad \frac{d^2\varphi}{dx^2} + \left(\frac{1}{x} - \beta x\right)\frac{d\varphi}{dx} - \left(\frac{1}{x^2} + \nu\beta\right)\varphi = 0.$$

The indicial equation at $x = 0$ in this case has roots $c_1 = 1$, $c_2 = -1$. Since we assumed existence of φ at $x = 0$, that is, that there was no hole at the center of the plate, we can make no use in this problem of the solution corresponding to c_2.

We reserve the details in obtaining the solution corresponding to $c_1 = 1$ for our first problem. Note, however, that we make even more striking use of the material in this book in Problem 2 below.

PROBLEMS

1. Solve equation (97) in the vicinity of $x = 0$, restricting yourself to the solution corresponding to $c_1 = 1$.

Hint: Check your result with that of Problem 2.

2. Use, in that order, the transformations $\varphi = x\psi$ and $\frac{\beta x^2}{2} = \sigma$ to reduce equation (97) to the form

$$\sigma\frac{d^2\psi}{d\sigma^2} + (2 - \sigma)\frac{d\psi}{d\sigma} - \tfrac{1}{2}(\nu + 1)\psi = 0.$$

---

* Actually the equation for φ is non-homogeneous, but, if inspection or trial does not lead to a particular integral, we may obtain one, in general, by applying our usual power-series methods. Thus the problem may be separated into parts of which the solution of equation (96) is one.

Thus show that, in terms of the Pochhammer-Barnes confluent hypergeometric function of Section 73,

$$\varphi_1(x) = xM\left(\frac{\nu+1}{2},\ 2,\ \frac{\beta x^2}{2}\right).$$

3. Returning to the equation

$$(96)\qquad \frac{d^2\varphi}{dx^2} + \left(\frac{1}{x} + \frac{d}{dx}\log y^3\right)\frac{d\varphi}{dx} - \left(\frac{1}{x^2} - \frac{\nu}{x}\frac{d}{dx}\log y^3\right)\varphi = 0,$$

define $t(x)$ by $y^3 = \exp(-t)$, where $t(0) = 0$. Obtain the differential equation for $\varphi(x)$ in terms of x and $t(x)$, and then subject that equation to the transformations $\varphi = x\psi$ and $x^2 = 2\sigma$. The function $t(x)$ is still not defined explicitly in terms of x.

Ans. $$\sigma\frac{d^2\psi}{d\sigma^2} + \left(2 - \sigma\frac{dt}{d\sigma}\right)\frac{d\psi}{d\sigma} - \tfrac{1}{2}(\nu+1)\frac{dt}{d\sigma}\psi = 0.$$

77. Another Choice for $y(x)$.

It is reasonable in the problem of the previous section to choose $y(x) = 1 + cx^k$, where c and k are constants and $k > 0$. Whether c is positive or negative we may, without introducing imaginary parameters, write this relation in at least one of the forms $y(x) = 1 - (\epsilon x)^k$ and $y(x) = 1 + (\epsilon x)^k$. Let us concentrate on $y(x) = 1 - (\epsilon x)^k$, since similar methods and results will apply to the other form for $y(x)$.

If we use $y(x) = 1 - (\epsilon x)^k$ in the equation

$$(96)\qquad \frac{d^2\varphi}{dx^2} + \left(\frac{1}{x} + \frac{d}{dx}\log y^3\right)\frac{d\varphi}{dx} - \left(\frac{1}{x^2} - \frac{\nu}{x}\frac{d}{dx}\log y^3\right)\varphi = 0,$$

we find that

$$\frac{d^2\varphi}{dx^2} + \left[\frac{1}{x} - \frac{3k\epsilon(\epsilon x)^{k-1}}{1-(\epsilon x)^k}\right]\frac{d\varphi}{dx} - \left[\frac{1}{x^2} + \frac{\nu}{x}\frac{3k\epsilon(\epsilon x)^{k-1}}{1-(\epsilon x)^k}\right]\varphi = 0.$$

Hence, we have the equation

$$x^2[1 - (\epsilon x)^k]\frac{d^2\varphi}{dx^2} + [1 - (\epsilon x)^k - 3k(\epsilon x)^k]\frac{d\varphi}{dx} - [1 - (\epsilon x)^k + 3k\nu(\epsilon x)^k]\varphi = 0.$$

Next we put $\epsilon x = z$ and arrive at the equation

$$z^2(1 - z^k)\frac{d^2\varphi}{dz^2} + (1 - z^k - 3kz^k)z\frac{d\varphi}{dz} - (1 - z^k + 3k\nu z^k)\varphi = 0.$$

Now, as in the previous section, let $\varphi = z\psi$. Our equation becomes

$$z^2(1 - z^k)\frac{d^2\psi}{dz^2} + 3(1 - z^k - kz^k)z\frac{d\psi}{dz} - 3k(\nu + 1)z^k\psi = 0.$$

Finally, in an attempt to simplify this equation we put $z^k = \sigma$. The details of the transformation are left for the student. The result is

$$k\sigma(1-\sigma)\frac{d^2\psi}{d\sigma^2} + [k+2-(4k+2)\sigma]\frac{d\psi}{d\sigma} - 3(\nu+1)\psi = 0,$$

or

$$\sigma(1-\sigma)\frac{d^2\psi}{d\sigma^2} + \left[\frac{k+2}{k} - \left(1+\frac{2+3k}{k}\right)\sigma\right]\frac{d\psi}{d\sigma} - \frac{3(\nu+1)}{k}\psi = 0.$$

This last equation is in precisely the form of the hypergeometric equation of Chapter VI. In the notation of that chapter, we have

$$\gamma = \frac{k+2}{2},$$

$$\alpha+\beta = \frac{2+3k}{k},$$

$$\alpha\beta = \frac{3(\nu+1)}{k}.$$

Thus, if α_1 and α_2 are the two roots of the equation

$$k\alpha^2 - (2+3k)\alpha + 3(\nu+1) = 0,$$

then our solution is

$$\psi = F\left(\alpha_1, \alpha_2, \frac{k+2}{2}, \sigma\right),$$

or in terms of the original problem,

$$\varphi_1(x) = \epsilon x F\left(\alpha_1, \alpha_2, \frac{k+2}{2}, \epsilon^k x^k\right).$$

78. Some Equations Drawn from Applications. In the problems below we present differential equations which arise in certain applications and indicate the type of applied problem for each. It will be noticed that throughout this chapter there is a preponderance of Bessel functions. This appears to be true in the literature on applications involving non-elementary functions, though it is somewhat less marked than is indicated by our specific choices.

PROBLEMS

1. In a problem involving the flow of a viscous fluid in a cylinder it is found necessary to solve the equation

$$x\frac{dy}{dx} = c^2x^2 - y^2,$$

where c is a real constant and a solution valid at $x = 0$ is desired. This Riccati equation may be transformed into a linear equation by the usual device. Put $y = \frac{xw'}{w}$, and solve the equation in w. Then obtain y.

Ans. $y = \frac{cxI_0'(cx)}{I_0(cx)}$.

2. In the study of the bending of a circular plate resting on an elastic foundation (Timoshenko, *Theory of Plates and Shells*, 1940, 275) we meet the equation

$$\left[\left(\mathrm{D}^2 + \frac{1}{x}\mathrm{D}\right)\left(\mathrm{D}^2 + \frac{1}{x}\mathrm{D}\right) + 1\right]z = 0,$$

where $\mathrm{D} \equiv \frac{d}{dx}$. Obtain the complete solution of this equation in the vicinity of $x = 0$. See Problem 2, Section 69.

3. In the study of the buckling of a column of variable cross section (see Timoshenko, *Theory of Elastic Stability*, 1936, 128–139) the following equation arises:

$$x^n\frac{d^2y}{dx^2} + k^2y = 0,$$

in which k is a positive constant and n is a positive integer. The value $n = 4$ is particularly important. It corresponds to a column which is a truncated pyramid or cone. In the above equation put $n = 4$, use the transformation $x = \frac{1}{\zeta}$, examine the invariant, and arrive at the solution, which is in terms of elementary functions.

4. The value $n = 2$ in the equation of Problem 3 is also of importance. Show that the solution is again elementary, and obtain it.

5. The value $n = 1$ in the equation of Problem 3 applies to a column whose cross section is a rectangle with one dimension constant, the other varying linearly. Solve the equation with $n = 1$.

6. Show that, in general, the equation of Problem 3 can be reduced to Bessel's equation, and perform the transformation. What exceptional case arises?

7. The equation

$$\frac{d^2\beta}{dw^2} + k^2w^2\beta = 0,$$

in which k is a real constant, arises in the study of the lateral buckling of a cantilever beam. See Timoshenko, *Theory of Elastic Stability*, 1936, 247. Solve this equation in the neighborhood of $w = 0$. Can you express the result in terms of Bessel functions?

8. In a problem on the buckling of prismatical bars under axial loads, we encounter the equation

$$\frac{d^3y}{dx^3} + k^2(l - x)\frac{dy}{dx} = 0,$$

in which k and l are positive constants. See Timoshenko, *Theory of Elastic Stability*, 1936, 115.

Following Timoshenko's presentation, put $z = \frac{2}{3}k(l - x)^{3/2}$ and then $u = \dfrac{dy}{dz}$. Write down the solution for u in terms of z and from it obtain y in terms of x.

9. In studying the behavior of human teeth under certain loads, Hay (see G. E. Hay, *Canadian Journal of Research*, A, **17**, 1939, 123–140) finds it necessary to solve the equations

$$(A) \qquad z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (4n^2 + z^2)w = z^3,$$

and

$$(B) \qquad z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (4n^2 + z^2)w = z^2,$$

in which n is a positive integer. Take the case $n = 3$, which is sufficiently representative, and obtain a particular solution valid in the vicinity of $z = 0$ for equation (A). Then write down the general solution of equation (A).

10. Obtain for equation (B) of Problem 9 a particular solution valid near $z = 0$ in the case $n = 3$. Note that the problem of obtaining a particular solution of equation (B) is somewhat different from the corresponding problem for equation (A). Discuss the reason for this.

79. Temperature in an Infinite Solid Cylinder. Let us obtain the temperature in an infinite solid circular cylinder of homogeneous material initially at a constant temperature and thereafter exposed to a different constant surface temperature. Mathematically this problem can be expressed as a boundary-value problem in the subject of partial differential equations. In these next few sections we shall be forced to restrict ourselves almost entirely to a purely formal presentation. For a thorough treatment of many points over which we shall pass the student should consult Churchill's *Fourier Series and Boundary Value Problems.*

Let us use cylindrical coordinates with r as the distance from the axis of our cylinder. The surface of the cylinder will be represented by $r = a$. Let θ represent temperature and t represent time. Then θ is a function of r and t, $\theta = \theta(r, t)$. The function $\theta(r, t)$ must satisfy the heat equation in which there occurs a constant, h^2, called the thermal diffusivity of the material. Let θ_0 be the initial constant temperature of the material and θ_1 the subsequent surface temperature. We wish to determine $\theta(r, t)$ from

$$(98)\qquad \begin{aligned} &\frac{\partial \theta}{\partial t} = h^2\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right), && \text{for}\quad 0 \leqq r < a, \quad 0 < t; \\ &\theta(r, 0) = \theta_0, && \text{for}\quad 0 \leqq r < a; \\ &\theta(a, t) = \theta_1, && \text{for}\quad 0 < t. \end{aligned}$$

We first simplify the above system by making the substitutions:

$$\begin{aligned} \theta &= (\theta_0 - \theta_1)u + \theta_1, \\ r &= ar_1, \\ t &= \frac{a^2 t_1}{h^2}. \end{aligned}$$

The boundary-value problem (98) then becomes

$$(99)\qquad \begin{aligned} &\frac{\partial u}{\partial t_1} = \frac{\partial^2 u}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial u}{\partial r_1}, && \text{for}\quad 0 \leqq r_1 < 1, \quad 0 < t_1; \\ &u(r_1, 0) = 1, && \text{for}\quad 0 \leqq r_1 < 1; \\ &u(1, t_1) = 0, && \text{for}\quad 0 < t_1. \end{aligned}$$

A common device in solving partial differential equations is to attempt to obtain solutions which are products of functions each of which contains only one of the independent variables. Let us put $u = F(t_1)w(r_1)$. Then the partial differential equation becomes

$$w\frac{dF}{dt_1} = F\left(\frac{d^2w}{dr_1^2} + \frac{1}{r_1}\frac{dw}{dr_1}\right),$$

or

$$\frac{1}{F}\frac{dF}{dt_1} = \frac{1}{w}\left(\frac{d^2w}{dr_1^2} + \frac{1}{r_1}\frac{dw}{dr_1}\right).$$

In this last equation the left member is a function of t_1 alone, the right member a function of r_1 alone. Hence each member of that equation must be constant. Thus we conclude that

$$\frac{1}{F}\frac{dF}{dt_1} = k,$$

$$\frac{1}{w}\left(\frac{d^2w}{dr_1^2} + \frac{1}{r_1}\frac{dw}{dr_1}\right) = k.$$

From the equation for F it follows that $F(t_1) = Ae^{kt_1}$, where A is constant. But we wish to have our solutions behave as $t_1 \to \infty$, so we restrict our constant k to real, negative values and put $k = -\alpha^2$. Thus $F(t_1) = Ae^{-\alpha^2 t_1}$. Now the equation for w becomes

$$\frac{d^2w}{dr_1^2} + \frac{1}{r_1}\frac{dw}{dr_1} + \alpha^2 w = 0.$$

Putting $\alpha r_1 = z$, we find the equation for w to be

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + w = 0,$$

which is Bessel's equation of index 0. Thus we have

$$w = BJ_0(\alpha r_1) + CY_0(\alpha r_1),$$

where B and C are arbitrary constants. Since $Y_0(\alpha r_1)$ has a singularity at $r_1 = 0$, we discard that part of our solution.

Finally, we combine our expressions for F and w, discard the redundant constant B, and write a solution,

$$u = Ae^{-\alpha^2 t_1}J_0(\alpha r_1),$$

of the differential equation of our problem. The boundary conditions,

$$\begin{aligned} u(r_1, 0) &= 1, \qquad \text{for} \qquad 0 \leqq r_1 < 1, \\ u(1, t_1) &= 0, \qquad \text{for} \qquad 0 < t_1, \end{aligned}$$

are not yet satisfied.

80. Continuation of the Solution. Let us now attempt to satisfy the boundary condition

$$u(1, t_1) = 0, \qquad \text{for} \qquad 0 < t_1.$$

We put $r_1 = 1$ and desire to make $Ae^{-\alpha^2 t_1}J_0(\alpha) = 0$ for all $t_1 > 0$. Our only chance here, since $A = 0$ would make it impossible to satisfy the other boundary condition, is to put

$$J_0(\alpha) = 0. \tag{100}$$

This equation determines our α, which was arbitrary up to now.

It can be proved that equation (100) has an infinity of real, positive roots: $\alpha_1, \alpha_2, \alpha_3, \cdots$, and no other roots except the, for us, trivial additions $-\alpha_1, -\alpha_2, -\alpha_3, \cdots$. We number the α_n in order of increasing magnitude. We may now associate with each of the α_n a corresponding solution $A_n e^{-\alpha_n^2 t_1} J_0(\alpha_n r_1)$ of the differential equation. Now only the A_n are left arbitrary.

Since the differential equation is linear, a sum of solutions is also a solution. Under proper conditions this may be extended to an infinite series of solutions. Hence we write

$$u = \sum_{n=1}^{\infty} A_n e^{-\alpha_n^2 t_1} J_0(\alpha_n r_1),$$

where the α_n are the successive positive zeros of $J_0(\alpha)$ and the A_n are constants yet to be determined.

In order to satisfy the one remaining condition,

$$u(r_1, 0) = 1, \qquad \text{for} \qquad 0 \leqq r_1 < 1,$$

we wish to determine the A_n so that

$$1 = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r_1).$$

It is necessary to prove that this expansion, namely, of 1 into such a series of Bessel functions, is possible. This we omit, merely noting that a large class of functions of r_1 can be so expanded. We turn instead to the determination of the A_n under the assumption that such an expansion exists.

81. An Orthogonality Property. We shall prove the orthogonality property,

$$\int_0^1 r_1 J_0(\alpha_m r_1) J_0(\alpha_n r_1)\, dr_1 = 0, \quad \text{if} \quad m \neq n. \tag{101}$$

From Bessel's equation of index zero we find that

$$\alpha_m r_1 J_0''(\alpha_m r_1) + J_0'(\alpha_m r_1) + \alpha_m r_1 J_0(\alpha_m r_1) = 0,$$
$$\alpha_n r_1 J_0''(\alpha_n r_1) + J_0'(\alpha_n r_1) + \alpha_n r_1 J_0(\alpha_n r_1) = 0,$$

where primes denote derivatives with respect to the indicated argument of the function. These equations may be written in more convenient forms,

$$\frac{d}{dr_1}[r_1 J_0'(\alpha_m r_1)] + \alpha_m r_1 J_0(\alpha_m r_1) = 0,$$

$$\frac{d}{dr_1}[r_1 J_0'(\alpha_n r_1)] + \alpha_n r_1 J_0(\alpha_n r_1) = 0.$$

Now we see a chance for expressing the product

$$r_1 J_0(\alpha_m r_1) J_0(\alpha_n r_1)$$

in terms of derivatives with respect to r_1 and thus performing the desired integration. Let us multiply the upper equation by $\alpha_m J_0(\alpha_n r_1)$, the lower by $\alpha_n J_0(\alpha_m r_1)$, and subtract one equation from the other. This yields the equation

$$J_0(\alpha_n r_1)\frac{d}{dr_1}[\alpha_m r_1 J_0'(\alpha_m r_1)] - J_0(\alpha_m r_1)\frac{d}{dr_1}[\alpha_n r_1 J_0'(\alpha_n r_1)] + (\alpha_m^2 - \alpha_n^2) r_1 J_0(\alpha_m r_1) J_0(\alpha_n r_1) = 0.$$

Hence

$$\begin{aligned}(\alpha_m^2 - \alpha_n^2) & r_1 J_0(\alpha_m r_1) J_0(\alpha_n r_1) \\ &= J_0(\alpha_m r_1)\frac{d}{dr_1}[\alpha_n r_1 J_0'(\alpha_n r_1)] - J_0(\alpha_n r_1)\frac{d}{dr_1}[\alpha_m r_1 J_0'(\alpha_m r_1)] \\ &= \frac{d}{dr_1}[\alpha_n r_1 J_0(\alpha_m r_1) J_0'(\alpha_n r_1) - \alpha_m r_1 J_0'(\alpha_m r_1) J_0(\alpha_n r_1)].\end{aligned}$$

We are now ready to perform the integration. We find that

$$\begin{aligned}(\alpha_m^2 - \alpha_n^2) & \int_0^1 r_1 J_0(\alpha_m r_1) J_0(\alpha_n r_1)\, dr_1 \\ &= [\alpha_n r_1 J_0(\alpha_m r_1) J_0'(\alpha_n r_1) - \alpha_m r_1 J_0'(\alpha_m r_1) J_0(\alpha_n r_1)]_0^1 \\ &= \alpha_n J_0(\alpha_m) J_0'(\alpha_n) - \alpha_m J_0'(\alpha_m) J_0(\alpha_n).\end{aligned}$$

But, $\alpha_m \neq \alpha_n$, $J_0(\alpha_m) = 0$, and $J_0(\alpha_n) = 0$. Hence we have proved

$$(101) \qquad \int_0^1 r_1 J_0(\alpha_m r_1) J_0(\alpha_n r_1)\, dr_1 = 0, \qquad \text{if} \qquad m \neq n.$$

82. Determination of the A_n. We now return to the expansion, assumed valid in $0 \leqq r_1 < 1$,

$$1 = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r_1).$$

From the above we have at once, assuming term-by-term integrability of the series,

$$\int_0^1 r_1 J_0(\alpha_m r_1)\, dr_1 = \sum_{n=1}^{\infty} A_n \int_0^1 r_1 J_0(\alpha_m r_1) J_0(\alpha_n r_1)\, dr_1.$$

In view of the orthogonality condition (101), each term of the right member vanishes except for the term with $n = m$. Hence

$$\int_0^1 r_1 J_0(\alpha_m r_1)\, dr_1 = A_m \int_0^1 r_1 J_0^2(\alpha_m r_1)\, dr_1.$$

The determination of the A_m is now reduced to the evaluation of two definite integrals.

First we note that

$$\int_0 r_1 J_0(\alpha_m r_1)\, dr_1 = -\frac{1}{\alpha_m}\int_0^1 \frac{d}{dr_1}\,[r_1 J_0'(\alpha_m r_1)]\, dr_1$$
$$= -\frac{1}{\alpha_m}\,[r_1 J_0'(\alpha_m r_1)]_0^1 = -\frac{J_0'(\alpha_m)}{\alpha_m}.$$

Let us denote the other integral as B for convenience. We have

$$B = \int_0^1 r_1 J_0^2(\alpha_m r_1)\, dr_1 = -\frac{1}{\alpha_m}\int_0^1 J_0(\alpha_m r_1)\,\frac{d}{dr_1}[r_1 J_0'(\alpha_m r_1)]\, dr_1.$$

Next we use integration by parts with the result

$$B = -\frac{1}{\alpha_m}\,[r_1 J_0(\alpha_m r_1) J_0'(\alpha_m r_1)]_0^1 + \int_0^1 r_1[J_0'(\alpha_m r_1)]^2\, dr_1$$
$$= \int_0^1 r_1[J_0'(\alpha_m r_1)]^2\, dr_1.$$

From Bessel's equation we find that

$$J_0'(\alpha_m r_1) = -\alpha_m r_1[J_0''(\alpha_m r_1) + J_0(\alpha_m r_1)].$$

Hence,

$$B = -\alpha_m \int_0^1 r_1^2[J_0'(\alpha_m r_1) J_0''(\alpha_m r_1) + J_0(\alpha_m r_1) J_0'(\alpha_m r_1)]\, dr_1$$
$$= -\tfrac{1}{2}\int_0^1 r_1^2\,\frac{d}{dr_1}\,\{[J_0'(\alpha_m r_1)]^2 + [J_0(\alpha_m r_1)]^2\}\, dr_1.$$

Once more we integrate by parts, this time obtaining

$$B = -\tfrac{1}{2}\left(r_1^2\{[J_0'(\alpha_m r_1)]^2 + [J_0(\alpha_m r_1)]^2\}\right)_0^1 + \int_0^1 r_1\{[J_0'(\alpha_m r_1)]^2 + [J_0(\alpha_m r_1)]^2\}\,dr_1,$$

or

$$B = -\tfrac{1}{2}[J_0'(\alpha_m)]^2 + \int_0^1 r_1[J_0'(\alpha_m r_1)]^2\,dr_1 + \int_0^1 r_1[J_0(\alpha_m r_1)]^2\,dr_1.$$

Each of the integrals on the right has the value B. Thus we have

$$B = -\tfrac{1}{2}[J_0'(\alpha_m)]^2 + 2B,$$

or

$$B = \tfrac{1}{2}[J_0'(\alpha_m)]^2.$$

Finally we return to the equation

$$\int_0^1 r_1 J_0(\alpha_m r_1)\,dr_1 = A_m \int_0^1 r_1 J_0^2(\alpha_m r_1)\,dr_1,$$

and use our evaluation of the two definite integrals to arrive at

$$-\frac{J_0'(\alpha_m)}{\alpha_m} = A_m \frac{[J_0'(\alpha_m)]^2}{2},$$

or

$$A_m = -\frac{2}{\alpha_m J_0'(\alpha_m)}.$$

It is easy to show directly from the series involved that

$$J_0'(z) = -J_1(z).$$

Hence

$$A_m = \frac{2}{\alpha_m J_1(\alpha_m)}.$$

We are able now to write the solution of our problem,

$$\begin{aligned}
&\frac{\partial u}{\partial t_1} = \frac{\partial^2 u}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial u}{\partial r_1}, &&\text{for} \quad 0 \leqq r_1 < 1, \quad 0 < t_1;\\
(99)\qquad &u(r_1, 0) = 1, &&\text{for} \quad 0 \leqq r_1 < 1;\\
&u(1, t_1) = 0, &&\text{for} \quad 0 < t_1.
\end{aligned}$$

That solution is

$$(102) \qquad u(r_1, t_1) = \sum_{n=1}^{\infty} \frac{2}{\alpha_n J_1(\alpha_n)} e^{-\alpha_n^2 t_1} J_0(\alpha_n r_1),$$

where the α_n are the successive positive zeros of $J_0(\alpha)$.

83. Remarks on Computations with the Above Solution. The solution exhibited in (102) above may, of course, be transformed into the solution of our original boundary-value problem (98) on page 199 by reversing the substitutions which led us to (99). That is unwise. When computing values of the series in (102), we are performing computations valid for all problems of the original type, independent of the parameters θ_0, θ_1, h^2, and a.

As an example of the procedure involved in direct computations, let us suppose that we are interested in the temperature at the axis of the cylinder at various times. From $r = 0$ we conclude that $r_1 = 0$ and therefore compute with the series

$$u(0, t_1) = \sum_{n=1}^{\infty} \frac{2}{\alpha_n J_1(\alpha_n)} e^{-\alpha_n^2 t_1}.$$

This is a function of one variable, t_1. Thus we may compute values of $u(0, t_1)$ and draw a single curve to represent that function. Thereafter that curve may be used for all problems of this type. That is, if we desire the temperature θ at a given time t, with parameters θ_0, θ_1, a, and h^2, given by the problem, we compute t_1 from

$$t_1 = \frac{h^2 t}{a^2}.$$

Then we refer to our previously drawn curve to find $u(0, t_1)$ and finally compute θ from the simple formula

$$\theta = (\theta_0 - \theta_1)u + \theta_1.$$

Some encouraging facts for actual computation follow. The series in (102) is rapidly convergent for all t_1 of practical value in most problems. Many α_n are tabulated,* and the $J_1(\alpha_n)$ are usually tabulated beside them.

---

* See, for example, the British Association for the Advancement of Science, *Mathematical Tables*, vol. 6, 1937, 171–173, where α_n and corresponding values of $J_1(\alpha_n)$ are tabulated to ten decimal places for $n = 1, 2, 3, \cdots, 150$.

Finally, if the initial temperature distribution in the cylinder is not constant, but a function of r, the method of the preceding sections goes through with very little change so long as the initial temperature function is a well-behaved one. Roughly speaking, the only change is that in the determination of A_m the integral

$$\int_0^1 r_1 J_0(\alpha_m r_1)\, dr_1$$

is replaced by one of the type

$$\int_0^1 r_1 f(r_1) J_0(\alpha_m r_1)\, dr_1,$$

where $f(r_1)$ is the equivalent, in the transformed problem, of the initial temperature function of the original problem.

84. Remarks on Other Boundary-Value Problems. The preceding application shows in a typical manner the modest role which the subject matter of this book plays in solving boundary-value problems in linear partial differential equations. Essentially what ordinarily occurs is that in attacking such a boundary-value problem the first step is to obtain a set of solutions of the partial differential equation. Most methods for accomplishing this aim reduce that phase of the problem to the solution of two or more problems in ordinary linear differential equations, which is where our subject takes over the burden.

Once the ordinary differential equations are solved, the difficult part of the problem begins. That is, the solutions obtained must be fitted together to satisfy the boundary conditions. Sections 79–83 give some hint of certain elementary methods for satisfying the boundary conditions. But it must be remembered that we proved the validity of almost none of our steps in that work. For the solution of certain simple problems of this type with proofs included the student is again advised to turn to Churchill's *Fourier Series and Boundary Value Problems.* More advanced tools for the same purpose will be exploited in a forthcoming book on modern operational mathematics in engineering, also by R. V. Churchill.

85. Problems Related to Laplace's Equation. An important example of the differential equations of applied mathematics is Laplace's equation, which in rectangular coordinates is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

This equation appears in problems of steady-state temperatures, gravitational potential, electric potential, hydrodynamics of ideal fluids, etc.

Suppose that we need to solve a boundary-value problem involving Laplace's equation. It may well be that values of the dependent variable are assigned on boundaries which will be given by constant values of some new independent variables. For example, a problem involving boundary values on a sphere would lead us to use spherical coordinates r, φ, and θ, and the following form of Laplace's equation:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0.$$

If we were attempting to solve a problem whose solution is independent of the coordinate φ, then the above equation could be reduced to

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) = 0.$$

If our problem also had $r = a$ as a boundary, it would be natural to attempt to find a solution of the form $u = F(r)w(\theta)$. This would lead at once to the equations

$$\frac{1}{F}\frac{d}{dr}\left(r^2 \frac{dF}{dr}\right) = k,$$

and

$$\frac{1}{w\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dw}{d\theta}\right) = -k.$$

The last equation can be written

$$\frac{d^2w}{d\theta^2} + \cot\theta\,\frac{dw}{d\theta} + kw = 0,$$

or, with $\cos\theta = z$, as

$$(1 - z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + kw = 0.$$

This is Legendre's equation with $k = n(n + 1)$.

In a similar manner varied problems involving Laplace's or other partial differential equations may lead us to familiar, or to new, ordinary differential equations.

Finally, we wish to point out that the present methods of attack on boundary-value problems in partial differential equations are seemingly restricted in their application to problems with comparatively simple geometric configurations for boundaries.

86. Polynomial Solutions of Linear Partial Differential Equations. We close with an example of a type of application which may be of use in either pure or applied mathematics. To revert to the application of Section 79, suppose that we have shown that

$$e^{-\alpha^2 t} J_0(\alpha r)$$

in which α is an arbitrary constant, is a solution of the linear partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

Then we may expand our solution into a power series in α. Since the function is a solution for all α, each coefficient of the power series must itself be a solution of the partial differential equation. Those coefficients will evidently be polynomials in r and t.

Since

$$e^{-\alpha^2 t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n \alpha^{2n}}{n!},$$

and

$$J_0(\alpha r) = \sum_{n=0}^{\infty} (-1)^n \frac{r^{2n}\alpha^{2n}}{2^{2n}(n!)^2},$$

we find that

$$\begin{aligned} e^{-\alpha^2 t} J_0(\alpha r) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{r^{2k}}{2^{2k}(k!)^2} (-1)^{n-k} \frac{t^{n-k}}{(n-k)!} \alpha^{2n} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \frac{r^{2k} t^{n-k}}{2^{2k}(k!)^2 (n-k)!} \alpha^{2n} \\ &= \sum_{n=0}^{\infty} (-1)^n \left[\sum_{k=0}^{n} \binom{n}{k} \frac{r^{2k} t^{n-k}}{2^{2k} k!} \right] \frac{\alpha^{2n}}{n!}. \end{aligned}$$

Hence the polynomials

$$V_n(r, t) = \sum_{k=0}^{n} \binom{n}{k} \frac{r^{2k} t^{n-k}}{2^{2k} k!}$$

are solutions of the partial differential equation under consideration.

For pure mathematics, possibly even for applied mathematics, it is more reasonable to consider a polynomial in one variable in this case. That is, since

$$V_n(r, t) = t^n \sum_{k=0}^{n} \binom{n}{k} \frac{[r^2/(4t)]^k}{k!} = t^n L_n\left(\frac{-r^2}{4t}\right),$$

the above might be looked at as leading us into a study of the Laguerre polynomial $L_n(z)$ of Section 72, page 189.

Evidently in the above work we may consider that $e^{-\alpha^2 t} J_0(\alpha r)$ is acting as a generating function for the polynomials $L_n(z)$. In pure mathematics we might introduce new polynomials in this way, though we do not need to start with the partial differential equation. Of course, we should not be astounded if, starting with commonly used partial differential equations, we are often led to the consideration of some equally well-known polynomials.

PROBLEMS

1. Use the process of this section to determine polynomial solutions of Laplace's equation in two rectangular coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

(*a*) From $u = e^{-\alpha y} \sin \alpha x$,
(*b*) From $u = e^{-\alpha y} \cos \alpha x$.

2. Obtain polynomial solutions of the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

3. Obtain polynomial solutions of the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

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